Transfinite diameter, Chebyshev constants and capacities in $\mathbb{C}^n$

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One of the fundamental achievements in complex analysis is the classical result of the first third of 20th century (Fekete, Szegö, Polya) which says that *for any compact set $K \subset \mathbb{C}$ the transfinite diameter $d(K)$, the Chebyshev constant $\tau(K)$ and the capacity $c(K)$ coincide*, although they are defined from very different points of view. Indeed, the transfinite diameter derives from a geometrical approach:

$$d(K) := \lim_{s\to\infty} d_s,\ d_s := \left(\max\left\{\prod_{i<j\leq s}\left|\zeta_i - \zeta_j\right| : \zeta_j \in K\right\}\right)^{2/s(s-1)}$$

as a *limit of geometrical means of some extremal distances between points of* K, the Chebyshev constant is defined in *terms of the least deviation of monic polynomials from zero*:

$$\tau(K) := \lim_{s\to\infty} \inf\left\{\max_{z\in K}\left|z^s + \sum_{j<s} c_j\ z^j\right| : c_j \in \mathbb{C},\ j = 1,\ldots,s-1\right\}^{1/s},$$

while the capacity appears from the *potential theory* considerations:

$$c(K) := \exp(-\lambda(K)),\ \lambda(K) := \lim_{z\to\infty}(g_K(z) - \ln|z|)$$

where $g_K(z)$ is the Green function for K with a logarithmic singularity at ∞.

For several complex variables, for a compact set K in $\mathbb{C}^n$, the transfinite diameter was introduced by F. Leja in 1957: $d(K) := \limsup_{s\to\infty} d_s(K)$, where $d_s(K)$ is determined analogously, in a sense, although much more complicated than in the one-dimensional case. He posed problem whether there exists an ordinary limit in his definition. This problem has been solved positively by the speaker in 1975, it was shown also that the *Leja transfinite diameter* coincides with, so-called, *main Chebyshev constant,* which is expressed as a continual geometric mean of *directional Chebyshev constants.* These results found recently their further development (Levenberg, Bloom, Bos et al) and applications in the arithmetic number theory (Rumely, Varley, Liu).

In our talk we discuss these and related topics.