A note on non-analytic functions

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We denote by $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disk in $\mathbb{C}$, and for a complex number $z = x + iy \in \mathbb{C}$, we denote by $\bar{z} = x - iy$, $|z| = \sqrt{x^2 + y^2}$ the complex conjugate, respectively the modulus of z. Let us consider the class of non-analytic functions $f(z)$ defined on the open unit disk $\mathbb{U}$ having a series expansion of the form

$$f(z, \bar{z}) = \sum_{n=1}^{\infty} f_n(z, \bar{z}) \qquad (z \in \mathbb{U}),$$

where $f_n = f_n(z, \bar{z})$ are complex functions defined for $z = x + iy \in \bar{\mathbb{U}}$, (real positive) homogeneous of degree n and satisfing a certain inequality on the boundary $\partial\mathbb{U}$.

Example 1 It is well known that the function $f(z)$ given by

$$f(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \qquad (z \in \mathbb{U})$$

is the extremal function for the class of convex functions in $\mathbb{U}$. If we consider the function $f(z, \bar{z})$ given by

$$f(z, \bar{z}) = \frac{\bar{z}}{1-\bar{z}} = \bar{z} + \sum_{n=2}^{\infty} \bar{z}^n \qquad (z \in \mathbb{U}),$$

then $f(z, \bar{z})$ is not analytic, but convex in $\mathbb{U}$. If we define

$$f(z, \bar{z}) = \frac{\bar{z}}{1-z} = \bar{z} + \sum_{n=2}^{\infty} |z|^2 z^{n-2} \qquad (z \in \mathbb{U}),$$

then $f(z, \bar{z})$ is not analytic and univalent in $\mathbb{U}$. Also

$$f(z, \bar{z}) = \frac{z}{1-\bar{z}} = z + \sum_{n=2}^{\infty} |z|^2 \bar{z}^{n-2} \qquad (z \in \mathbb{U})$$

is not analytic and univalent in $\mathbb{U}$.

Example 2 Koebe function $f(z)$ defined by

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{n=2}^{\infty} n z^n \qquad (z \in \mathbb{U})$$

is the extremal function for the class of starlike functions in $\mathbb{U}$. If we consider the function $f(z, \bar{z})$ given by

$$f(z, \bar{z}) = \frac{\bar{z}}{(1-\bar{z})^2} = \bar{z} + \sum_{n=2}^{\infty} n \bar{z}^n \qquad (z \in \mathbb{U}),$$

then $f(z,\bar{z})$ is not analytic in $\mathbb{U}$, but it maps $\mathbb{U}$ on to the starlike domain. Let us consider the function

$$f(z,\bar{z}) = \frac{\bar{z}}{(1-z)^2} = \bar{z} + \sum_{n=2}^{\infty} n|z|^2 z^{n-2} \qquad (z \in \mathbb{U}).$$

Then $f(z,\bar{z})$ is not analytic and univalent in $\mathbb{U}$. Also the function

$$f(z,\bar{z}) = \frac{z}{(1-\bar{z})^2} = z + \sum_{n=2}^{\infty} n|z|^2 \bar{z}^{n-2} \qquad (z \in \mathbb{U})$$

is not analytic and univalent in $\mathbb{U}$.

Theorem 1 *Let $f(z,\bar{z})$ defined for $z \in \mathbb{U}$ have a series expansion of the form*

$$f(z,\bar{z}) = \sum_{n=1}^{\infty} f_n(z,\bar{z}), \quad z \in \mathbb{U},$$

where $f_n(z,\bar{z})$ are functions of $z \in \bar{\mathbb{U}}$ satisfying

$$f_n(rz, r\bar{z}) = r^n f_n(z,\bar{z}),$$

for all $z \in \bar{\mathbb{U}}$ and real numbers $r > 0$ for which $rz \in \bar{\mathbb{U}}$, $n = 1,2,3,\ldots$. If for some $\theta \in [0,2\pi)$ we have

$$\sum_{n=2}^{\infty} n \left| f_n(e^{i\theta}, e^{-i\theta}) \right| \leq \left| f_1(e^{i\theta}, e^{-i\theta}) \right| \neq 0,$$

then $|f(z,\bar{z})|$ is an increasing function of $|z|$ on $\arg z = \theta$, that is

$$|f(z_1,\overline{z_1})| < |f(z_2,\overline{z_2})|,$$

for any $z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\theta} \in \mathbb{U}$ with $0 < r_1 < r_2 < 1$.
In particular, if the condition holds for all $\theta \in [0,2\pi)$, then $|f(z)|$ is radially increasing in the whole open unit disk $\mathbb{U}$, and it cannot therefore attain its maximum at an interior point of $\mathbb{U}$.

Theorem 2 *Let $f(z,\bar{z})$ defined for $z \in \mathbb{U}$ have a series expansion of the form*

$$f(z,\bar{z}) = \sum_{n=1}^{\infty} f_n(z,\bar{z}) \qquad (z \in \mathbb{U}),$$

where $f_n(z,\bar{z})$ are functions of $z \in \bar{\mathbb{U}}$ satisfying

$$f_n(rz, r\bar{z}) = r^n f_n(z,\bar{z}),$$

for all $z \in \bar{\mathbb{U}}$ and real numbers $r > 0$ for which $rz \in \bar{\mathbb{U}}$, $n = 1,2,3,\ldots$ and

$$f_1(z,\bar{z}) \neq 0$$

for all $z \in \bar{\mathbb{U}}$.
If for some $\theta \in [0,2\pi)$ we have

$$\left| f_n(e^{i\theta}, e^{-i\theta}) \right| \leq \frac{1}{n^{2+\alpha}} \frac{\min_{\theta \in [0,2\pi)} \left| f_1(e^{i\theta}, e^{-i\theta}) \right|}{\zeta(1+a) - 1} \qquad (n = 2,3,4,\ldots;\ a > 0),$$

then $|f(z,\bar{z})|$ is an increasing function of $|z|$ on $\arg z = \theta$, that is

$$|f(z_1,\overline{z_1})| < |f(z_2,\overline{z_2})|,$$

for any $z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\theta} \in \mathbb{U}$ with $0 < r_1 < r_2 < 1$.