Hypercyclicity of Weighted Backward Shifts on Spaces of Real Analytic Functions

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Outline

- Introduction
- Multipliers on the spaces of real analytic functions
- Density arguments
- Conditions on hypercyclicity of the weighted backward shifts
- Hypercyclicity of the usual backward shift

Definition

An operator T on a topological vector space X is called *hypercyclic* if there is some $x \in X$ such that the set

$$\{x, Tx, T^2x, \cdots, T^nx, \cdots\}$$

is dense in X.

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is dense in X.

The set $\{x, Tx, T^2x, \cdots\}$ is called the *orbit* of x under T.

(Birkhoff transitivity theorem) An operator T on a separable Fréchet space X is hypercyclic, if and only if, it is *topologically transitive*, that is, for any pair of nonempty open subsets U, V of X, there exists some $n \in \mathbb{N}$ such that

$$T^n(U) \cap V \neq \emptyset.$$

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Remark: Any hypercyclic operator on a general topological vector space is topologically transitive, however the converse may not be true.

(Bonet 2000) A topologically transitive linear operator on an arbitrary locally convex space need not be hypercyclic.

Examples of hypercyclic operators:
(Birkhoff 1929) The *translation operators*

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(MacLane 1952) The *differentiation operator* $D : f \mapsto f'$ on $H(\mathbb{C})$,
(Rolewicz 1962) The multiples of the *backward shift*

$$\lambda B(x_n)_{n\in\mathbb{N}} = (\lambda x_{n+1})_{n\in\mathbb{N}}$$

for any λ with $|\lambda| > 1$, on the sequence spaces l_p, $1 \leq p < \infty$, or c_0.

Let X be a Fréchet sequence space with canonical unit sequences e_n. Then, for a sequence of nonzero scalars $\omega = (\omega_n)_{n\in\mathbb{N}}$, the operator $B_\omega : X \to X$ defined by

$$B_\omega e_n = \omega_n e_{n-1},\ \ n \geq 1, \quad e_0 = 0,$$

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is called *a weighted backward shift* on X.

Theorem (Grosse-Erdmann 2000)

The operator $B_\omega : X \to X$, acting on a Fréchet sequence space X in which $(e_n)_{n\in\mathbb{N}}$ is a basis, is hypercyclic, if and only if, there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$\left(\prod_{\nu=1}^{n_k} \omega_\nu\right)^{-1} e_{n_k} \to 0$$

in X as $k \to \infty$.

Let $A(\Omega)$ denote the space of all complex-valued real analytic functions on an open set Ω in $\mathbb{R}$, that is, every function in $A(\Omega)$ develops into a Taylor series at each point of Ω.

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Topology on $A(\Omega)$

- *Projective limit topology*

$$A(\Omega) = \mathrm{proj}_{N\in\mathbb{N}} H(K_N) = \mathrm{proj}_{N\in\mathbb{N}}\ \mathrm{ind}_{n\in\mathbb{N}}\ H^\infty(U_{N,n}),$$

where $(K_N)_{N\in\mathbb{N}}$ is a compact increasing exhaustion of Ω, and $(U_{N,n})_{n\in\mathbb{N}}$ are fundamental sequences of neighborhoods of K_N for each N.

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- *Inductive limit topology*

$$A(\Omega) = \mathrm{ind}\ H(U),$$

  where the inductive limit is taken over all complex neighborhoods of Ω.

- **(Martineau 1966)** These topologies are equivalent.
- $A(\Omega)$ is a complete, separable, ultrabornological, nuclear, reflexive space.
- The closed graph theorem and the open mapping theorem hold in $A(\Omega)$.
- **(Domański, Vogt 2000)** $A(\Omega)$ has no Schauder basis.

Definition

Given a sequence of nonzero scalars $\omega = (\omega_n)_{n\in\mathbb{N}}$, a linear continuous operator

$$B_\omega : A(\Omega) \to A(\Omega),$$

that sends

- the monomials x^n to $\omega_n x^{n-1}$ for all $n \in \mathbb{N}$,
- the unit function to the zero function,

is called a *weighted backward shift* with the weight sequence ω.

Multipliers on $A(\Omega)$

A linear continuous operator

$$M : A(\Omega) \to A(\Omega)$$

is called a *multiplier* whenever every monomial is an eigenvector. The corresponding sequence of eigenvalues $(m_n)_{n\in\mathbb{N}}$ is called the *multiplier sequence* for M.

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Theorem (Domański, Langenbruch 2012)

Any multiplier sequence $(m_n)_{n\in\mathbb{N}}$ is a sequence of Laurent coefficients of some function g which is holomorphic at infinity, that is,

$$g(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}}.$$

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Proof If $B_\omega : A(\Omega) \to A(\Omega)$ is a weighted backward shift with the weight sequence $\omega = (\omega_n)_{n\in\mathbb{N}}$, then the map $M : A(\Omega) \to A(\Omega)$ defined by

$$M(f)(x) = B_\omega(xf(x)),\ f \in A(\Omega),\ x \in \Omega,$$

is a multiplier with the multiplier sequence ω.

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is a multiplier with the multiplier sequence ω.
Similarly, if $M : A(\Omega) \to A(\Omega)$ is a multiplier, then the map $T : A(\Omega) \to A(\Omega)$ defined by

$$T(f)(x) = M\left(\frac{f(x) - f(0)}{x}\right),\ f \in A(\Omega),\ x \in \Omega,$$

is a weighted backward shift.

Density Arguments

Let $H(\mathbb{C})$ denote the space of entire functions, and $H(\{0\})$ denote the space of germs of holomorphic functions at zero.

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If B_ω is a weighted backward shift on $A(\Omega)$, then B_ω is also a weighted backward shift on $H(\{0\})$ and $H(\mathbb{C})$.

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Lemma

If B_ω is a weighted backward shift on $A(\Omega)$, then B_ω is also a weighted backward shift on $H(\{0\})$ and $H(\mathbb{C})$.

Proof Let $\omega = (\omega_n)$ be a weight sequence. Then, ω is also a multiplier sequence, and it can be represented as a sequence of Laurent coefficients of some function which is holomorphic at infinity. Hence, $\exists r > 0$ such that

$$\sup_n |\omega_n| r^n < \infty.$$

We can then show that the maps $B_\omega : H(\{0\}) \to H(\{0\})$ and $B_\omega : H(\mathbb{C}) \to H(\mathbb{C})$ are well-defined linear continuous maps.

Density Arguments

As $H(\mathbb{C})$ is dense in $A(\Omega)$, and $A(\Omega)$ is dense in $H(\{0\})$ whenever $0 \in \Omega$, we have the following observation.

Lemma

- *If B_ω is hypercyclic on $H(\mathbb{C})$, then it is also hypercyclic on $A(\Omega)$.*
- *If B_ω is hypercyclic on $A(\Omega)$, then it is also hypercyclic on $H(\{0\})$.*

Proposition

For an open set Ω in $\mathbb{R}$ with $0 \in \Omega$, and a weighted backward shift $B_\omega : A(\Omega) \to A(\Omega)$ with the weight sequence $\omega = (\omega_n)_{n\in\mathbb{N}}$,

(a) *if there is an increasing sequence (n_k) of positive integers such that for all $R > 0$,*

$$\lim_{k\to\infty} \left(\left(\prod_{\nu=1}^{n_k} \omega_\nu \right)^{-1} R^{n_k} \right) = 0,$$

then B_ω is hypercyclic on $A(\Omega)$,

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then B_ω is hypercyclic on $A(\Omega)$,

(b) *if B_ω is hypercyclic on $A(\Omega)$, then there exist an increasing sequence (n_k) of positive integers and $R > 0$ such that*

$$\lim_{k\to\infty} \left(\left(\prod_{\nu=1}^{n_k} \omega_\nu \right)^{-1} R^{n_k} \right) = 0.$$

Proof

(a) Let B_ω be a weighted backward shift on $A(\Omega)$. Then, it is also a weighted backward shift on $H(\mathbb{C})$. Since $H(\mathbb{C})$ is a Fréchet sequence space, the given condition implies that B_ω is hypercyclic on $H(\mathbb{C})$. Hence, B_ω is hypercyclic on $A(\Omega)$.

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(b) Let B_ω be hypercyclic on $A(\Omega)$. Then, it is also hypercyclic on $H(\{0\})$.
The space $H(\{0\})$ is isomorphic to the nuclear Köthe co-echelon space $k_p(V)$, $1 \leq p \leq \infty$, where

$$k_p(V) = \text{ind}_{n\to} l_p(v_n)$$

with $V = (v_{nk})$, $v_{nk} = e^{-kn}$.

(Bierstedt, Meise, Summers 1982) For $1 \leq p < \infty$, $k_p(V)$ is topologically isomorphic to the space

$$K_p(\bar{V}) = \text{proj}_{\leftarrow \bar{v} \in \bar{V}} l_p(\bar{v})$$
$$= \left\{ x = (x_k) : \forall \bar{v} \in \bar{V} \ \|x\|_{\bar{v}} = \left(\sum_{k=1}^{\infty} |x_k|^p \bar{v}_k^p \right)^{1/p} \right\},$$

where $\bar{V} = \{\bar{v} = (\bar{v}_k) \in \mathbb{R}_+^{\mathbb{N}} : \sup_k \frac{\bar{v}_k}{v_{nk}} < \infty \ \forall n \in \mathbb{N}\}$.

(Bierstedt, Meise, Summers 1982) For $1 \leq p < \infty$, $k_p(V)$ is topologically isomorphic to the space

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Therefore, $H(\{0\})$ is topologically isomorphic to $K_p(\bar{V})$, and B_ω is hypercyclic on $K_p(\bar{V})$ by our assumption.

Since $B_\omega : K_p(\bar{V}) \to K_p(\bar{V})$ is continuous, given $\bar{v}^{(0)} \in \bar{V}$, we can obtain inductively that for every $n \in \mathbb{N}$, there exists $\bar{v}^{(n)} \in \bar{V}$ and constant C_n so that

$$\|B_\omega x\|_{\bar{v}^{(n-1)}} \leq C_n \|x\|_{\bar{v}^{(n)}}, \quad x \in K_p(\bar{V}).$$

Hence, B_ω is continuous on $K_p(\bar{V})$ equipped with the topology given by the sequence of norms $(\|\cdot\|_{\bar{v}^{(n)}})_{n\in\mathbb{N}}$.

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By completing this space, we obtain a Fréchet space X with the following properties:

- X is isomorphic to the Köthe sequence space $\lambda_p((v^{(n)})_{n\in\mathbb{N}})$,
- X contains $K_p(\bar{V})$ continuously and densely,
- B_ω is a weighted backward shift on X.

Since B_ω is hypercyclic on $K_p(\bar{V})$, and $K_p(\bar{V})$ is dense in X, B_ω is also hypercyclic on X. As X is a Fréchet sequence space, there is an increasing sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that

$$\left(\prod_{\nu=1}^{n_k} \omega_\nu\right)^{-1} e_{n_k} \to 0$$

in X as $k \to \infty$.

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Then, we can show that there exists $R > 0$ satisfying

$$\lim_{k\to\infty} \left(\left(\prod_{\nu=1}^{n_k} \omega_\nu\right)^{-1} R^{n_k}\right) = 0.$$

Problem

Clearly, there are weight sequences satisfying the condition

$$\lim_{k\to\infty}\left(\left(\prod_{\nu=1}^{n_k}\omega_\nu\right)^{-1}R^{n_k}\right)=0$$

for some $R > 0$, but not all $R > 0$.

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Example The usual backward shift on $A(\Omega)$, that is, $\omega=(\omega_n)$ where $\omega_n=1$ for all $n\in\mathbb{N}$.

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Example The usual backward shift on $A(\Omega)$, that is, $\omega=(\omega_n)$ where $\omega_n=1$ for all $n\in\mathbb{N}$.

Question Is the usual backward shift on $A(\Omega)$ hypercyclic?

Hypercyclicity of the Usual Backward Shift on $A(\Omega)$

Theorem

The usual backward shift on $A(\mathbb{R})$ is hypercyclic .

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Proof The usual backward shift $B : A(\mathbb{R}) \to A(\mathbb{R})$, where $B(x^n) = x^{n-1}$ for all $n \in \mathbb{N}$ and $B(\mathbf{1}) = \mathbf{0}$, coincides with the function

$$T(f)(x) = \frac{f(x) - f(0)}{x}$$

on polynomials. Since the polynomials are dense in $A(\mathbb{R})$, we have $B = T$.

Hypercyclicity of the Usual Backward Shift on $A(\Omega)$

If we take the strip

$$S = \{z \in \mathbb{C} : |\mathrm{Im}\ z| < 1/2\},$$

then $H(S)$ is dense in $A(\mathbb{R})$, so it is enough to show that T is hypercyclic on $H(S)$. For this purpose, we need the following criterion.

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Godefroy-Shapiro criterion

Let T be an operator on a separable Fréchet space X. If the subspaces

$$X_0 := \mathrm{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| < 1\},$$
$$Y_0 := \mathrm{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| > 1\}$$

are dense in X, then T is hypercyclic.

Hypercyclicity of the Usual Backward Shift on $A(\Omega)$

Solving the equation $Tf = \lambda f$, we can observe that for any $\zeta \in \hat{\mathbb{C}} \setminus S$, the function

$$f_\zeta(z) = \frac{1}{\zeta - z}$$

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Using a variation of Runge's Theorem and the Grothendieck - Köthe - Silva duality, we show that

$$\mathrm{span}\{f_\zeta : \zeta \in \hat{\mathbb{C}} \backslash S\}$$

are dense in $H(S)$ for the separate cases $|\zeta| < 1$ and $|\zeta| > 1$.

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Therefore, by the Godefroy-Shapiro criterion, T is hypercyclic on $H(S)$, which implies that T is hypercyclic on $A(\mathbb{R})$.

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