Tame Fréchet Spaces

Buket Can Bahadır

Middle East Technical University

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A linear map $T : E \to F$, where E and F are Fréchet Spaces with increasing seminorm systems, is continuous if for each $k \in \mathbb{N}$ there exists $j \in \mathbb{N}$ and $C_k > 0$ such that

$$\| Tx \|_k \leq C_k \| x \|_j, \ \forall x \in E.$$

We define π_T, the *characteristic of continuity map* of the linear operator T as:

$$\pi_T(k) \doteq \inf\{\sigma \in \mathbb{N} : \sup_{\|x\|_\sigma \leq 1} \| Tx \|_k < \infty\}.$$

Let:

$$\| T \|_{k,r} \doteq \sup_{\|x\|_r \leq 1} \| Tx \|_k \ .$$

Then clearly $\pi_T(k) = \inf_{\sigma \in \mathbb{N}} \| T \|_{k,\sigma}$.

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Definition

We say that the pair (E, F), where E and F are Fréchet spaces, is called a *tame pair*, and denote $(E, F) \in \mathfrak{T}$, if there exists an increasing function $S : \mathbb{N} \to \mathbb{N}$ such that for all $T \in L(E, F)$ there exists $k_0 \in \mathbb{N}$ and a constant C_k with

$$\| Tx \|_k \leq C_k \| x \|_{S(k)} \quad \forall x \in E, \forall k \geq k_0.$$

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Let $S : \mathbb{N} \to \mathbb{N}$ be an increasing function. Let $T \in L(E, F)$, where E and F are Fréchet spaces.
We say that T is S-*tame* if there exists $k_0 \in \mathbb{N}$ such that $\forall k \geq k_0$, $\exists C_k$ with

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Given a nondecreasing function $\varphi : \mathbb{N} \to \mathbb{N}$, for $T \in L(E,F)$, we define:

$$L_\varphi(E,F) \doteq \{T \in L(E,F) : \forall k\, \exists C\, \forall x : \| Tx \|_k \leq C \| x \|_{\varphi(k)}\} \quad (1)$$
$$= \{T \in L(E,F) : \| T \|_{k,\varphi(k)} < \infty\, \forall k\}. \quad (2)$$

Then clearly, $L_\varphi(E,F)$ is a Fréchet Space with seminorm system $\| \,.\, \|_{k,\varphi(k)}$, $k \in \mathbb{N}$.

With this notation, we have:

Theorem (Piszczek)

The following are equivalent:

i. (E, F) *is a tame pair*
ii. *There exists an increasing function* $S : \mathbb{N} \to \mathbb{N}$*, tending to infinity such that for any other increasing function* $\varphi : \mathbb{N} \to \mathbb{N}$ *tending to infinity we have:*

$$\exists k \forall m \geq k \, \exists n, C_m \, \forall T \in L_\varphi(E, F) :$$

$$\max_{k \leq l \leq m} \| T \|_{S(l),l} \leq C_m \max_{1 \leq p \leq n} \| T \|_{\varphi(p),p}$$

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If $(\lambda(A), \lambda(B)) \in \mathfrak{T}$, where $\lambda(A)$ and $\lambda(B)$ are Köthe Spaces with corresponding matrices (a_{ip}) and $(b_{\nu q})$ by applying this theorem to all one dimensional operators $| \, e_\nu \otimes e_i' \, |$, we get, for each $\pi : \mathbb{N} \to \mathbb{N}$ there exists $\alpha \in \mathbb{N}$ such that for each $r \in \mathbb{N}$ there are $q_0 \in \mathbb{N}$ and $C > 0$ with

$$\frac{b_{\nu r}}{a_{iS_\alpha(r)}} \leq C \sup_{q \leq q_0} \{ \frac{b_{\nu q}}{a_{i\pi(q)}} \}, \quad \forall i, \nu.$$

On the other hand, if the above inequality holds, for any $T \in L(\lambda(A), \lambda(B))$, we can find $S_\alpha : \mathbb{N} \to \mathbb{N}$'s, such that the tameness criteria is satisfied.

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Let $A = (a_{ip})$ and $B = (b_{\nu q})$ be Köthe matrices. The following are equivalent:

i) $(\lambda^p(A), \lambda^q(B)) \in \mathfrak{T}$ *for every pair* (p, q), $1 \leq p, q \leq \infty$, *where* $p = 1$ *or* $q = \infty$

ii) $(\lambda^p(A), \lambda^q(B)) \in \mathfrak{T}$ *for some pair* (p, q), $1 \leq p, q \leq \infty$, *where* $p = 1$ *or* $q = \infty$

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Dragilev and Nurlu proved that the existence of an continuous linear unbounded operator between nuclear Köthe Spaces implies the existence of a continuous unbounded quasi-diagonal operator between them. Djakov and Ramanujan sharpened this result by omitting the nuclearity condition. We can state a similar result about tameness of two Köthe Spaces.

Theorem

$L(\lambda^p(A), \lambda^q(B))$ is tame for every pair (p, t), $1 \leq p, q \leq \infty$, where $p = 1$ or $q = \infty$ if and only if the family of continuous quasidiagonal operators from $\lambda^p(A)$ to $\lambda^q(B)$ is tame.

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Power series spaces are defined as:

$$\Lambda_r^p(\alpha) \doteq \{x \in \mathbb{K}^{\mathbb{N}} : \| x \|_k = \Big(\sum_{j \in \mathbb{N}} e^{p r_k \alpha_j} \mid x_j \mid^p \Big)^{1/p} < \infty \forall k\}, \quad 1 \leq p < \infty,$$

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where α is a nondecreasing sequence of nonnegative scalars which tends to infinity and $r_k \nearrow r$. If $r = 0$, the space is of finite type, and if $r \to \infty$ it is of infinite type.

The sequence α is called *stable* if

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Vogt proved that finite type power series spaces are always tame, and Dubinsky and Vogt characterized the tameness in infinite type power series spaces.

The following table summarizes the characterization of tameness between power series spaces.

	$\Lambda_0(\beta)$ stable	$\Lambda_0(\beta)$ nonstable	$\Lambda_\infty(\beta)$ stable	$\Lambda_\infty(\beta)$ nonstable
$\Lambda_0(\alpha)$ stable	Tame	Tame	Tame	Tame
$\Lambda_0(\alpha)$ nonstable	Tame	Tame	Tame	Tame
$\Lambda_\infty(\alpha)$ stable	Non-tame	Non-tame	Tame iff bounded	Tame iff bounded
$\Lambda_\infty(\alpha)$ nonstable	Non-tame	Tame with conditions	Tame iff bounded	Tame with conditions

Theorem

$(\Lambda_0(\alpha), \Lambda_\infty(\beta))$ is a tame pair for any α, β.

Proof.

Since any $T \in L(\Lambda_0(\alpha), \Lambda_\infty(\beta))$ is compact (Zakhariuta), $(\Lambda_0(\alpha), \Lambda_\infty(\beta))$ is tame. □

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Theorem (Nyberg)

The following conditions are equivalent.

(i) *Every operator from $\Lambda_\infty(\alpha)$ to $\Lambda_0(\beta)$ is bounded,*

(ii) *$(\Lambda_\infty(\alpha), \Lambda_0(\beta))$ is tame,*

(iii) *$(\Lambda_\infty(\alpha), \Lambda_0(\beta))$ is linearly tame,*

(iv) *$M_{\beta\alpha}$; the set of finite limit points of $(\beta_i/\alpha_j)_{i,j\in\mathbb{N}}$ is bounded.*

When α is stable, all linear operators are bounded if and only if $\Lambda_0^p(\beta))$ has the property (LB_∞) (Vogt). This implies that $\Lambda_0^p(\beta))$ has the property (DN), which is impossible. When β is stable, the pair is tame if and only if $\Lambda_\infty(\alpha)$ is $(\overline{\Omega})$ (Piszcek), which is a contradiction.

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If α or β is stable then $(\Lambda_\infty(\alpha), \Lambda_0(\beta))$ is not tame.

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Example

Let β and γ be stable.
Consider $\gamma_j = 2^{j^2}$ and $c_{jn} = \exp\{nj^n\gamma_j\}$. Since $\Lambda_1^p(\gamma)$ is nuclear $\lambda_1^p(\gamma) = \Lambda_\infty^p(\gamma)$. Then $(\lambda(C), \Lambda_1^p(\gamma))$ is tame but not bounded.

Similarly, for $\beta_j = 2^{j^2}$ and $c_{jn} = \exp\{-\frac{1}{n}j^{1/n}\gamma_j\}$, $(\Lambda_1^p(\beta), (\lambda(C))$ is tame but not bounded.

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When $\alpha = \beta$, Vogt gave some refined results about tameness.

Given power series sequences α, β, we find increasing sequences t_n, s_n, and define another power series sequence γ by letting $\gamma_{t_n} = \alpha_n$, $\gamma_{s_n} = \beta_n$, and $\{(\gamma_n)_{n\in\mathbb{N}}\} = \{(\gamma_{t_n})_{n\in\mathbb{N}}\} \cup \{(\gamma_{s_n})_{n\in\mathbb{N}}\}$.

We define the operator $T_1 : \Lambda_\infty(\gamma) \to \Lambda_\infty(\alpha)$ by $T_1((x_n)_{n\in\mathbb{N}}) \doteq (x_{t_n})_{n\in\mathbb{N}}$,

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Theorem

The range of every tame operator $T \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ *has a basis.*

Corollary

If in addition $\Lambda_\infty(\beta)$ *is nuclear, then* $Range(T)$ *has an absolute basis.*

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If in this case, $\Lambda_\infty(\beta)$ *is nuclear, and* $Range(T)$ *is closed, Then* $Range(T)$ *is isomorphic to a closed subspace of* s.

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$T \in L(\Lambda_r(\alpha) \times \Lambda_s(\beta))$ has a matrix representation:

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} : (x, y) \to (T_{11}x + T_{12}y, T_{21}x + T_{22}y),$$

where $T_{11} \in L(\Lambda_r(\alpha), \Lambda_r(\alpha))$, $T_{12} \in L(\Lambda_r(\alpha), \Lambda_s(\beta))$, $T_{21} \in L(\Lambda_s(\beta), \Lambda_r(\alpha))$, $T_{22} \in L(\Lambda_s(\beta), \Lambda_s(\beta))$.

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We have the following about cartesian product of power series spaces:

- *The cartesian product $\Lambda_0(\alpha) \times \Lambda_0(\beta)$ is always tame.*
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Let E, F and G be Fréchet Spaces. An operator $T \in L(E, F)$ *factors over* G, if T can be expressed as a composition of two operators $R \in L(E, G)$ and $Q \in L(G, F)$.

Definition

When E, F and G are Fréchet Spaces, we will say that the triple (E, G, F) has the *tame factorization property*, and denote it by $(E, G, F) \in \mathfrak{T}\mathfrak{F}$ if there exists a nondecreasing map $S : \mathbb{N} \to \mathbb{N}$ such that

$$\pi_T(k) \leq S(k)$$

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Terzioglu and Zakhariuta studied the bounded factorization property of Fréchet spaces, and characterized them.

We want to characterize the pairs of Fréche Spaces, for which the continuity characteristic maps of all linear continuous operators that can be linearly and continuously factored over a third Fréchet spaces can be estimated by a nondecreasing map $S : \mathbb{N} \to \mathbb{N}$.

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We have the following:

Theorem

For Frećhet Spaces E, F and G, (E, G, F) has the tame factorization property iff and only if there exists nondecreasing $S_\alpha : \mathbb{N} \to \mathbb{N}$ such that for each $\pi : \mathbb{N} \to \mathbb{N}$, there exists α such that for every $r \in \mathbb{N}$ there is some $q_0 \in \mathbb{N}$ and $C > 0$ with

$$\| RQ \|_{r,S_\alpha(r)} \leq C \sup_{q \leq q_0} \{\| R \|_{q,\pi(q)}\} \sup_{q \leq q_0} \{\| Q \|_{q,\pi(q)}\}$$

for every $R \in L(G, F)$ and $Q \in L(E, G)$.

For Köthe spaces we get more refined results:

Theorem

For Köthe Spaces $\tilde{\lambda}(A)$, $\tilde{\lambda}(B)$ and $\tilde{\lambda}(C)$, where $\tilde{\lambda}$ is either λ or λ_0^∞, $(\tilde{\lambda}(A), \tilde{\lambda}(B), \tilde{\lambda}(C)) \in \mathfrak{T}\mathfrak{F}$ if and only if there exists nondecreasing $S_\alpha : \mathbb{N} \to \mathbb{N}$ such that for each $\pi : \mathbb{N} \to \mathbb{N}$, there exists α such that for every $r \in \mathbb{N}$ there is some $q_0 \in \mathbb{N}$ and $C > 0$ with

$$\frac{c_{\nu r}}{a_{i S_\alpha(r)}} \leq C \sup_{q \leq q_0}\left\{\frac{b_{jq}}{a_{i\pi(q)}}\right\} \sup_{q \leq q_0}\left\{\frac{c_{\nu q}}{b_{j\pi(q)}}\right\},$$

holds for all $i, j, \nu \in \mathbb{N}$

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$\lambda(A), \lambda(B), \lambda(C) \in \mathfrak{T}\mathfrak{F}$ if and only if $\mathcal{A} = \{T = RQ : R \in L(\lambda(B), \lambda(C))$ and $Q \in L(\lambda(A), \lambda(B))$ are quasidiagonal$\}$ is tame.

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We have the following characterization about factorization of power series spaces:

Theorem

If β and γ are stable, then

- $(\Lambda^p_\infty(\beta), \lambda(B)\hat{\otimes}_\pi\lambda(C), \Lambda^q_\infty(\gamma)) \in \mathfrak{TF}$ *if and only if* $(\Lambda^p_\infty(\beta), \lambda(B)\hat{\otimes}_\pi\lambda(C), \Lambda^q_\infty(\gamma)) \in \mathfrak{BF}$,
- $(\Lambda^p_0(\beta), \lambda(B)\hat{\otimes}_\pi\lambda(C), \Lambda^q_0(\gamma)) \in \mathfrak{TF}$ *if* $\lambda(B) \in (\underline{DN})$, $\lambda(C)$ *nuclear and has property* $(\overline{\Omega})$,
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