

Bergman Kernel in Convex Domains

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$\Omega \subset \mathbb{C}^n$, $w \in \Omega$

$$K_{\Omega}(w) = \sup\{|f(w)|^2 : f \in \mathcal{O}(\Omega), \int_{\Omega} |f|^2 d\lambda \leq 1\}$$

(Bergman kernel on the diagonal)

$$G_w(z) = G_{\Omega}(z, w)$$

$$= \sup\{u(z) : u \in PSH^-(\Omega) : \overline{\lim}_{z \rightarrow w} (u(z) - \log |z - w|) < \infty\}$$

(pluricomplex Green function)

Theorem 0 Assume Ω is pseudoconvex in \mathbb{C}^n . Then for $w \in \Omega$ and $t \leq 0$

$$K_{\Omega}(w) \geq \frac{1}{e^{-2nt} \lambda(\{G_w < t\})}.$$

Optimal constant: “=” if $\Omega = B(w, r)$.

Proof 1 Using Donnelly-Fefferman's estimate for $\bar{\partial}$ one can prove

$$K_{\Omega}(w) \geq \frac{1}{c(n, t)\lambda(\{G_w < t\})}, \quad (1)$$

where

$$c(n, t) = \left(1 + \frac{C}{Ei(-nt)}\right)^2, \quad Ei(a) = \int_a^{\infty} \frac{ds}{se^s}$$

(B. 2005). Now use the tensor power trick: $\tilde{\Omega} = \Omega \times \cdots \times \Omega \subset \mathbb{C}^{nm}$, $\tilde{w} = (w, \dots, w)$ for $m \gg 0$. Then

$$K_{\tilde{\Omega}}(\tilde{w}) = (K_{\Omega}(w))^m, \quad \lambda(\{G_{\tilde{w}} < t\}) = (\lambda(\{G_w < t\}))^m,$$

and by (1) for $\tilde{\Omega}$

$$K_{\Omega}(w) \geq \frac{1}{c(nm, t)^{1/m}\lambda(\{G_w < t\})}.$$

But $\lim_{m \rightarrow \infty} c(nm, t)^{1/m} = e^{-2nt}$. □

Proof 2 (Lempert) By Berndtsson's result on log-(pluri)subharmonicity of the Bergman kernel for sections of a pseudoconvex domain it follows that $\log K_{\{G_w < t\}}(w)$ is convex for $t \in (-\infty, 0]$. Therefore

$$t \longmapsto 2nt + \log K_{\{G_w < t\}}(w)$$

is convex and bounded, hence non-decreasing. It follows that

$$K_{\Omega}(w) \geq e^{2nt} K_{\{G_w < t\}}(w) \geq \frac{e^{2nt}}{\lambda(\{G_w < t\})}. \quad \square$$

Berndtsson-Lempert: This method can be improved to show the Ohsawa-Takegoshi extension theorem with optimal constant.

Theorem 0 Assume Ω is pseudoconvex in \mathbb{C}^n . Then for $w \in \Omega$ and $t \leq 0$

$$K_{\Omega}(w) \geq \frac{1}{e^{-2nt} \lambda(\{G_w < t\})}.$$

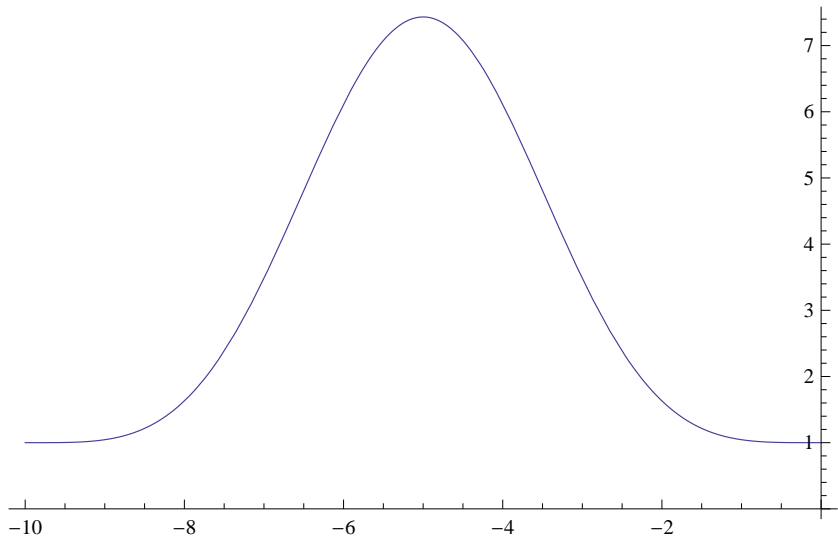
What happens when $t \rightarrow -\infty$? For $n = 1$ Theorem 0 immediately gives:

Theorem (Suiata conjecture) For a domain $\Omega \subset \mathbb{C}$ one has

$$K_{\Omega}(w) \geq c_{\Omega}(w)^2/\pi, \quad w \in \Omega, \quad (2)$$

where $c_{\Omega}(w) = \exp\left(\lim_{z \rightarrow w} (G_{\Omega}(z, w) - \log|z - w|)\right)$
(logarithmic capacity of $\mathbb{C} \setminus \Omega$ w.r.t. w).

Theorem (Guan-Zhou) Equality holds in (2) iff $\Omega \simeq \Delta \setminus F$, where Δ is the unit disk and F a closed polar subset.



$\frac{\pi K_\Omega}{c_\Omega^2}$ for $\Omega = \{e^{-5} < |z| < 1\}$ as a function of $2 \log |w|$

What happens with $e^{-2nt}\lambda(\{G_w < t\})$ as $t \rightarrow -\infty$ for arbitrary n ? For convex Ω using Lempert's theory one can get

Proposition If Ω is bounded, smooth and strongly convex in \mathbb{C}^n then for $w \in \Omega$

$$\lim_{t \rightarrow -\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I_\Omega^K(w)),$$

where $I_\Omega^K(w) = \{\varphi'(0) : \varphi \in \mathcal{O}(\Delta, \Omega), \varphi(0) = w\}$ (Kobayashi indicatrix).

Corollary If $\Omega \subset \mathbb{C}^n$ is convex then

$$K_\Omega(w) \geq \frac{1}{\lambda(I_\Omega^K(w))}, \quad w \in \Omega.$$

For general Ω one can prove

Theorem If Ω is bounded and hyperconvex in \mathbb{C}^n and $w \in \Omega$ then

$$\lim_{t \rightarrow -\infty} e^{-2nt}\lambda(\{G_w < t\}) = \lambda(I_\Omega^A(w)),$$

where $I_\Omega^A(w) = \{X \in \mathbb{C}^n : \overline{\lim}_{\zeta \rightarrow 0} (G_w(w + \zeta X) - \log |\zeta|) \leq 0\}$
(Azukawa indicatrix)

Corollary (SCV version of the Suita conjecture) If $\Omega \subset \mathbb{C}^n$ is pseudoconvex and $w \in \Omega$ then

$$K_{\Omega}(w) \geq \frac{1}{\lambda(I_{\Omega}^A(w))}.$$

Remark 1. For $n = 1$ one has $\lambda(I_{\Omega}^A(w)) = \pi/c_{\Omega}(w)^2$.

2. If Ω is convex then $I_{\Omega}^A(w) = I_{\Omega}^K(w)$.

Conjecture For Ω pseudoconvex and $w \in \Omega$ the function

$$t \mapsto e^{-2nt} \lambda(\{G_w < t\})$$

is non-decreasing in t .

It would easily follow if we knew that the function

$$t \mapsto \log \lambda(\{G_w < t\})$$

is convex on $(-\infty, 0]$. Fornæss however constructed a counterexample to this (already for $n = 1$).

Theorem The conjecture is true for $n = 1$.

Proof It is enough to prove that $f'(t) \geq 0$ where

$$f(t) := \log \lambda(\{G_w < t\}) - 2t$$

and t is a regular value of G_w . By the co-area formula

$$\lambda(\{G_w < t\}) = \int_{-\infty}^t \int_{\{G_w=s\}} \frac{d\sigma}{|\nabla G_w|} ds$$

and therefore

$$f'(t) = \frac{\int_{\{G_w=t\}} \frac{d\sigma}{|\nabla G_w|}}{\lambda(\{G_w < t\})} - 2.$$

By the Schwarz inequality

$$\int_{\{G_w=t\}} \frac{d\sigma}{|\nabla G_w|} \geq \frac{(\sigma(\{G_w = t\}))^2}{\int_{\{G_w=t\}} |\nabla G_w| d\sigma} = \frac{(\sigma(\{G_w = t\}))^2}{2\pi}.$$

The isoperimetric inequality gives

$$(\sigma(\{G_w = t\}))^2 \geq 4\pi\lambda(\{G_w < t\})$$

and we obtain $f'(t) \geq 0$. □

The conjecture for arbitrary n is equivalent to the following *pluricomplex isoperimetric inequality* for smooth strongly pseudoconvex Ω

$$\int_{\partial\Omega} \frac{d\sigma}{|\nabla G_w|} \geq 2n\lambda(\Omega).$$

The conjecture also turns out to be closely related to the problem of symmetrization of the complex Monge-Ampère equation.

What about the corresponding upper bound in the Suita conjecture?
Not true in general:

Proposition Let $\Omega = \{r < |z| < 1\}$. Then

$$\frac{K_{\Omega}(\sqrt{r})}{(c_{\Omega}(\sqrt{r}))^2} \geq \frac{-2 \log r}{\pi^3}.$$

It would be interesting to find an upper bound of the Bergman kernel for domains in \mathbb{C} in terms of logarithmic capacity which would in particular imply the \Rightarrow part in the well known equivalence (due to Carleson)

$$K_{\Omega} > 0 \Leftrightarrow c_{\Omega} > 0$$

$(c_{\Omega}^2 \leq \pi K_{\Omega}$ being a quantitative version of \Leftarrow).

The upper bound for the Bergman kernel holds for convex domains:

Theorem For a convex Ω and $w \in \Omega$ set

$$F_{\Omega}(w) := (K_{\Omega}(w)\lambda(I_{\Omega}^K(w)))^{1/n}.$$

Then $F_{\Omega}(w) \leq 4$.

Sketch of proof Denote $I := \text{int } I_{\Omega}^K(w)$ and assume that $w = 0$. One can show that $I \subset 2\Omega$. Then

$$K_{\Omega}(0)\lambda(I) \leq K_{I/2}(0)\lambda(I) = \frac{\lambda(I)}{\lambda(I/2)} = 4^n. \quad \square$$

If Ω is in addition symmetric w.r.t. w then $F_{\Omega}(w) \leq 16/\pi^2 = 1.621\dots$

Remark The proof of the optimal lower bound $F_{\Omega} \geq 1$ used $\bar{\partial}$.
The proof of the (probably) non-optimal upper bound $F_{\Omega} \leq 4$ is much more elementary!

For convex domains

$$F_{\Omega}(w) = (\lambda(I_{\Omega}(w))K_{\Omega}(w))^{1/n}$$

is a biholomorphically invariant function satisfying $1 \leq F_{\Omega} \leq 4$.

- Find an example with $F_{\Omega} \neq 1$.
- What are the properties of the function $w \mapsto \lambda(I_{\Omega}(w))$?
- What is the optimal upper bound for F_{Ω} ?

For the rest of the talk we will restrict ourselves to studying convex complex ellipsoids in \mathbb{C}^2 :

$$\mathcal{E}(p, q) = \{z \in \mathbb{C}^2 : |z_1|^{2p} + |z_2|^{2q} < 1\}, \quad p, q \geq 1/2.$$

Caution It will get technical!

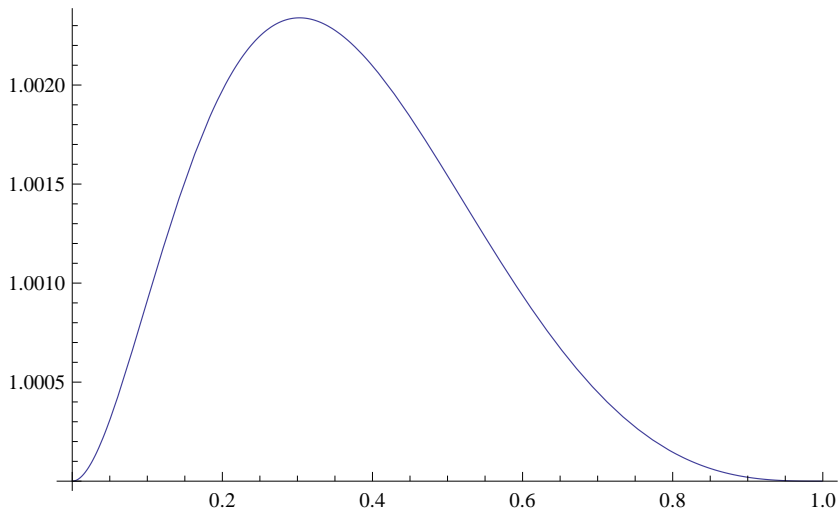
Blank-Fan-Klein-Krantz-Ma-Pang (1992) found implicit formulas for the Kobayashi function of $\mathcal{E}(m, 1)$. They can be made explicit for $m = 1/2$. Using this one can prove

Theorem For $\Omega = \{|z_1| + |z_2|^2 < 1\}$ and $b \in [0, 1)$ one has

$$\lambda(l_\Omega((b, 0))) = \frac{\pi^2}{3}(1-b)^3(1+3b+3b^2-b^3)$$

and

$$\lambda(l_\Omega((b, 0)))K_\Omega((b, 0)) = 1 + \frac{(1-b)^3 b^2}{3(1+b)^3}.$$



$F_{\Omega}((b, 0))$ for $\Omega = \{|z_1| + |z_2|^2 < 1\}$

Although the Kobayashi function of $\mathcal{E}(m, 1)$ is given by implicit formulas, it turns out that the volume of the Kobayashi indicatrix can be computed explicitly:

Theorem For $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$, $m \geq 1/2$, and $b \in [0, 1)$ one has

$$\begin{aligned} & \lambda(I_{\Omega}((b, 0))) \\ &= \pi^2 \left[-\frac{m-1}{2m(3m-2)(3m-1)} b^{6m+2} - \frac{3(m-1)}{2m(m-2)(m+1)} b^{2m+2} \right. \\ & \quad \left. + \frac{m}{2(m-2)(3m-2)} b^6 + \frac{3m}{3m-1} b^4 - \frac{4m-1}{2m} b^2 + \frac{m}{m+1} \right]. \end{aligned}$$

For $m = 2/3$

$$\lambda(I_{\Omega}((b, 0))) = \frac{\pi^2}{80} \left(-65b^6 + 40b^6 \log b + 160b^4 - 27b^{10/3} - 100b^2 + 32 \right),$$

and $m = 2$

$$\lambda(I_{\Omega}((b, 0))) = \frac{\pi^2}{240} \left(-3b^{14} - 25b^6 - 120b^6 \log b + 288b^4 - 420b^2 + 160 \right).$$

About the proof Main tool: Jarnicki-Pflug-Zeinstra (1993) formula for geodesics in convex complex ellipsoids. If

$$\mathbb{C} \supset U \ni z \mapsto (f(z), g(z)) \in \partial I$$

is a parametrization of an S^1 -invariant portion of ∂I then the volume of the corresponding part of I is given by

$$\frac{\pi}{2} \int_U |H(z)| d\lambda(z), \quad (3)$$

where

$$H = |f|^2(|g_{\bar{z}}|^2 - |g_z|^2) + |g|^2(|f_{\bar{z}}|^2 - |f_z|^2) + 2\operatorname{Re}(f\bar{g}(\bar{f}_z g_z - \bar{f}_{\bar{z}} g_{\bar{z}})).$$

Both H and the integral (3) are computed with the help of *Mathematica*. The same method is used for computations in other ellipsoids.

For $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ the formula for the Bergman kernel is well known:

$$K_{\Omega}(w) = \frac{1}{\pi^2} (1 - |w_2|^2)^{1/m-2} \frac{(1/m + 1)(1 - |w_2|^2)^{1/m} + (1/m - 1)|w_1|^2}{((1 - |w_2|^2)^{1/m} - |w_1|^2)^3},$$

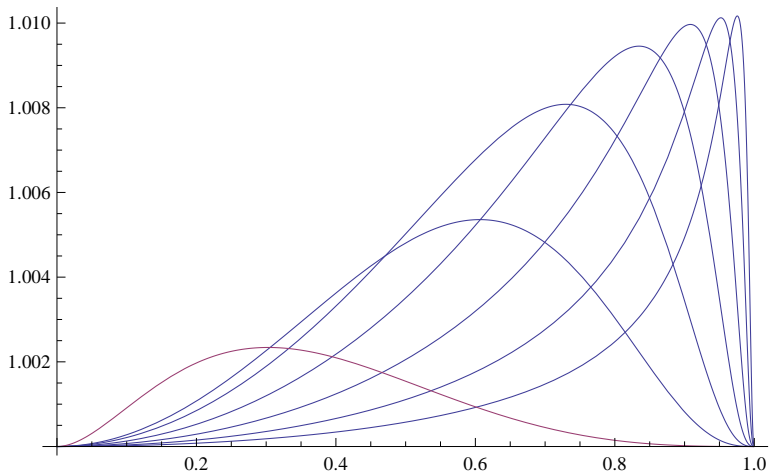
so that

$$K_{\Omega}((b, 0)) = \frac{m + 1 + (1 - m)b^2}{\pi^2 m(1 - b^2)^3}.$$

Since for $t \in \mathbb{R}$ and $a \in \Delta$ the mapping

$$\Omega \ni z \mapsto \left(e^{it} \frac{(1 - |a|^2)^{1/2m}}{(1 - \bar{a}z_2)^{1/m}} z_1, \frac{z_2 - a}{1 - \bar{a}z_2} \right)$$

is a holomorphic automorphism of Ω , $F_{\Omega}((b, 0))$ for $b \in [0, 1)$ attains all values of F_{Ω} in Ω .



$F_{\Omega}((b, 0))$ in $\Omega = \{|z_1|^{2m} + |z_2|^2 < 1\}$ for $m = 1/2, 4, 8, 16, 32, 64, 128$

$$\sup_{0 < b < 1} F_{\Omega}((b, 0)) \rightarrow 1.010182\dots \text{ as } m \rightarrow \infty$$

(highest value of F_{Ω} obtained so far in arbitrary dimension)

Theorem For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1)$ one has

$$\lambda(l_\Omega((b, 0))) = \frac{\pi^2}{6}(1-b)^4((1-b)^4 + 8b)$$

and

$$\lambda(l_\Omega((b, 0)))K_\Omega((b, 0)) = 1 + b^2 \frac{(1-b)^4}{(1+b)^4}.$$

The Bergman kernel for this ellipsoid was found by Hahn-Pflug (1988):

$$K_\Omega(w) = \frac{2}{\pi^2} \cdot \frac{3(1-|w|^2)^2(1+|w|^2) + 4|w_1|^2|w_2|^2(5-3|w|^2)}{((1-|w|^2)^2 - 4|w_1|^2|w_2|^2)^3},$$

so that

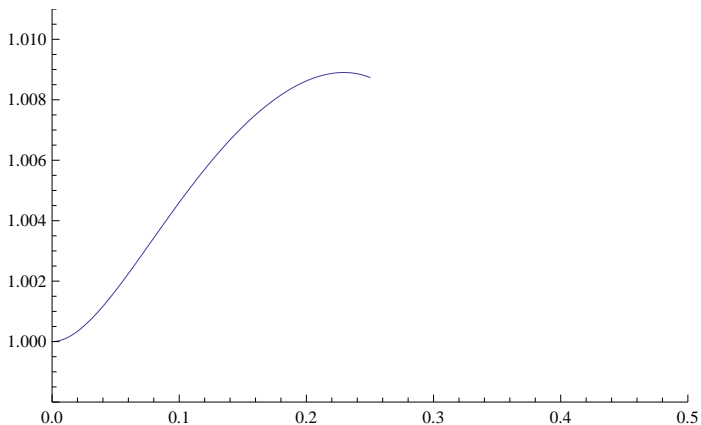
$$K_\Omega((b, 0)) = \frac{6(1+b^2)}{\pi^2(1-b^2)^4}.$$

In all examples so far the function $w \mapsto \lambda(l_\Omega(w))$ is analytic. Is it true in general?

Theorem For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

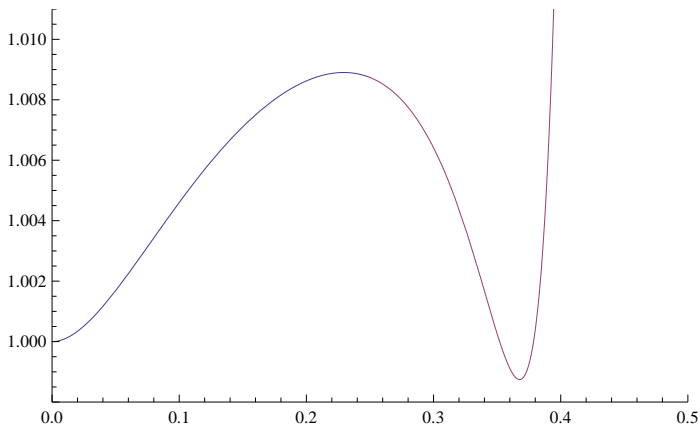
$$\lambda(l_{\Omega}((b, b))) = \frac{\pi^2}{6} (30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1).$$

Since $K_{\Omega}((b, b)) = \frac{2(3 - 6b^2 + 8b^4)}{\pi^2(1 - 4b^2)^3}$, we get the following picture:



$F_{\Omega}((b, b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/4]$

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$F_{\Omega}((b, b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/4]$

By either of the estimates $1 \leq F_{\Omega} \leq 4$, the function $b \mapsto F_{\Omega}((b, b))$ cannot be analytic on $(0, 1/2)$!

Theorem For $\Omega = \{|z_1| + |z_2| < 1\}$ and $b \in [0, 1/4]$ one has

$$\lambda(I_{\Omega}((b, b))) = \frac{\pi^2}{6} (30b^8 - 64b^7 + 80b^6 - 80b^5 + 76b^4 - 16b^3 - 8b^2 + 1).$$

For $b \in [1/4, 1/2)$

$$\begin{aligned} \lambda(I_{\Omega}((b, b))) &= \frac{2\pi^2 b(1-2b)^3 (-2b^3 + 3b^2 - 6b + 4)}{3(1-b)^2} \\ &+ \frac{\pi (30b^{10} - 124b^9 + 238b^8 - 176b^7 - 260b^6 + 424b^5 - 76b^4 - 144b^3 + 89b^2 - 18b + 1)}{6(1-b)^2} \\ &\quad \times \arccos \left(-1 + \frac{4b-1}{2b^2} \right) \\ &+ \frac{\pi(1-2b) (-180b^7 + 444b^6 - 554b^5 + 754b^4 - 1214b^3 + 922b^2 - 305b + 37)}{72(1-b)} \sqrt{4b-1} \\ &+ \frac{4\pi b(1-2b)^4 (7b^2 + 2b - 2)}{3(1-b)^2} \arctan \sqrt{4b-1} \\ &+ \frac{4\pi b^2(1-2b)^4 (2-b)}{(1-b)^2} \arctan \frac{1-3b}{(1-b)\sqrt{4b-1}}. \end{aligned}$$

By $\chi_-(b)$, resp. $\chi_+(b)$, denote $\lambda(l_\Omega((b, b)))$ for $b \leq 1/4$, resp. $b \geq 1/4$.
Then at $b = 1/4$

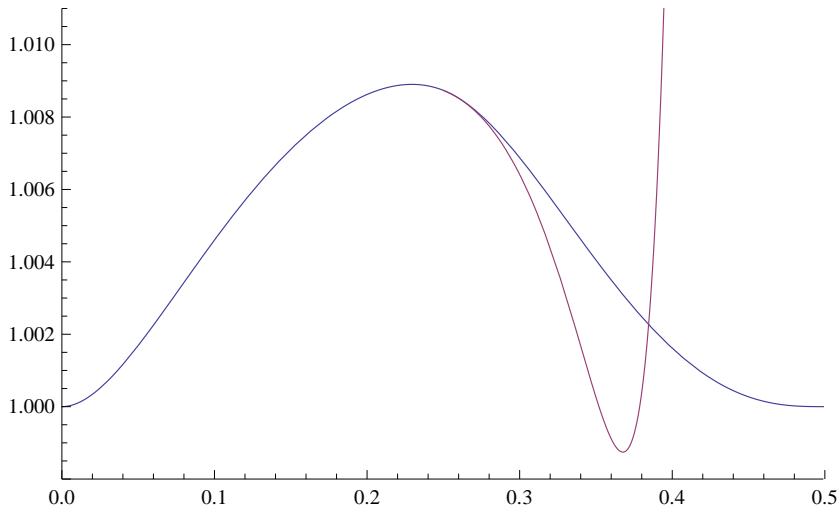
$$\chi_- = \chi_+ = \frac{15887}{196608}\pi^2, \quad \chi'_- = \chi'_+ = -\frac{3521}{6144}\pi^2,$$

$$\chi''_- = \chi''_+ = -\frac{215}{1536}\pi^2, \quad \chi^{(3)}_- = \chi^{(3)}_+ = \frac{1785}{64}\pi^2,$$

but

$$\chi_-^{(4)} = \frac{1549}{16}\pi^2, \quad \chi_+^{(4)} = \infty.$$

Corollary For $\Omega = \{|z_1| + |z_2| < 1\}$ the function $w \mapsto \lambda(l_\Omega(w))$ is not $C^{3,1}$ at $w = (1/4, 1/4)$.



$F_{\Omega}((b, b))$ in $\Omega = \{|z_1| + |z_2| < 1\}$ for $b \in [0, 1/2)$

Thank you!