A NEW CLASS OF COHERENT STATES WITH MEIXNER-POLLACZEK POLYNOMIALS FOR THE GOL'DMAN-KRIVCHENKOV HAMILTONIAN

ZOUHAÏR MOUAYN

ABSTRACT. A class of generalized coherent states with a new type of the identity resolution are constructed by replacing the labeling parameter $z^n/\sqrt{n!}$ of the canonical coherent states by Meixner-Pollaczek polynomials with specific parameters. The constructed coherent states belong to the state Hilbert space of the Gol'dman-Krivchenkov Hamiltonian.

1 INTRODUCTION

Coherent states are mathematical tools which provide a close connection between classical and quantum formalisms and have essentially many definitions. In general, coherent states are a specific overcomplete family of vectors in the Hilbert space of the problem that describes the quantum phenomena and solves the identity of this Hilbert space [1].

The canonical coherent states for the harmonic oscillator have long been known and their properties have frequently been taken as models for defining the notion of coherent states for other models [2, 5].

In this paper, we are concerned with the model of the Gol'dman-Krivchenkov Hamiltonian [6] acting on the Hilbert space of square integrable functions on the positive real half-line. We, precisely construct a family of coherent states labeled by points of the whole real line, depending on some parameters and belonging to the state Hilbert space of this Hamiltonian. To achieve this, we will adopt a formalism of canonical coherent states when written as superpositions of the harmonic oscillator number states. That is, we present our generalized coherent states as a superposition of an orthonormal basis of the state Hilbert space of the Gol'dman-Krivchenkov Hamiltonian, whose identity is solved by a new way. We choose the Meixner-Pollaczek polynomials to play the role of coefficients in this superposition. This choice enables us to present the constructed states in a closed form.

The paper is organized as follows. In Section 2, we recall briefly some needed spectral properties of the Gol'dman-Krivchenkov Hamiltonian. Section 3 is devoted to the coherent states formalism we will be using. This formalism is applied in Section 4 so as to construct a family of coherent states in the state Hilbert space of the Hamiltonian we are dealing with. In Section 5 we conclude with a summary.

2 THE GOL'DMAN-KRIVCHENKOV HAMILTONIAN

An anharmonic potential that can be used to calculate the vibrational energies of a diatomic molecules has the form

$$V_{\varrho,\kappa_0}(\xi) := \varrho \left(\frac{\xi}{\kappa_0} - \frac{\xi_0}{\xi} \right)^2 \tag{2.1}$$

where $\kappa_0 > 0$ denotes the equilibrium bond length which is the distance between the diatomic nuclei, and $\varrho > 0$ with $F = \varrho \kappa_0^{-2}$ represents a constant force. The associated stationary Schrödinger equation reads

$$-\frac{d^2}{d\xi^2}\psi(\xi) + \varrho \left(\frac{\xi}{\kappa_0} - \frac{\kappa_0}{\xi} \right)^2 \psi(\xi) = \lambda\psi(\xi), \tag{2.2}$$

with $\psi(0) = 0$, namely ψ satisfies the Dirichlet boundary condition. It is an exactly solvable equation. Indeed, according to [7, p. 11288], the energy spectrum is given by

$$\lambda_m^{\varrho,\kappa_0} := 4\kappa_0^{-1}\sqrt{\varrho}\left(m + \frac{1}{2} + \frac{1}{4}\left(\sqrt{1 + 4\varrho\kappa_0^2} - 2\kappa_0\sqrt{\varrho}\right)\right), \quad m = 0,1,2,\cdots \quad (2.3)$$

whereas the wave functions of the exact solutions of Eq. (2.2) take the form

$$< \xi \mid \psi_m^{\varrho,\kappa_0} > \propto \xi^q \exp\left(-\frac{\sqrt{\varrho}}{2\kappa_0}\xi^2\right) {}_1F_1\left(-m, q+1, \frac{\sqrt{\varrho}}{2\kappa_0}\xi^2\right), \quad (2.4)$$

where $q = \frac{1}{2}\left(1 + \sqrt{1 + 4\varrho\kappa_0^2}\right)$ and ${}_1F_1(.)$ denotes the confluente hypergeometric function which can also be expressed in terms of Laguerre polynomials as [8, p. 240]:

$$_1F_1\left(-m, q+1, u\right) = \frac{m!}{(q)_m}L_m^{(q)}(u) \quad (2.5)$$

in terms of the Pochhammer symbols

$$(a)_0 = 1, (a)_m = a(a+1)\cdots(a+m-1) = \frac{\Gamma(a+m)}{\Gamma(a)}, \quad m = 1,2,\cdots. \quad (2.6)$$

To simplify the notation, we introduce the new parameters $\alpha := \varrho\kappa_0^2$ and $\beta := \kappa_0^{-1}\sqrt{\varrho}$, and thereby the Hamiltonian in Eq. (2.2) takes the form

$$\Delta_{\alpha,\beta} := -\frac{d^2}{d\xi^2} + \beta^2\xi^2 + \frac{\alpha}{\xi^2}, \quad \xi \in \mathbb{R}_+, \beta, \alpha > 0 \quad (2.7)$$

called Gol'dman-Krivchenvov Hamiltonian ([7, p. 11288]). Its spectrum in the Hilbert space $L^2(\mathbb{R}_+, d\xi)$ reduces to a discrete part consisting of eigenvalues of the form ([9, pp. 9-10]):

$$\lambda_m^{\gamma,\beta} := 2\beta(2m + \gamma(\alpha)), \gamma = \gamma(\alpha) = 1 + \frac{1}{2}\sqrt{1 + 4\alpha} \quad, m = 0,1,2,\cdots, \quad (2.8)$$

and wavefunctions of the corresponding normalized eigenfunctions are given by

$$< \xi \mid \psi_m^{\gamma,\beta} > := \left(\frac{2\beta^\gamma m!}{\Gamma(\gamma+m)}\right)^{\frac{1}{2}}\xi^{\gamma-\frac{1}{2}}e^{-\frac{1}{2}\beta\xi^2}L_m^{(\gamma-1)}\left(\beta\xi^2\right), \quad m = 0,1,2,\cdots \quad (2.9)$$

The set of functions in (2.9) constitutes a complete orthonormal basis for the Hilbert space $L^2(\mathbb{R}_+, d\xi)$.

Remark 2.1. We should note that the eigenvalues in (2.8) together with their eigenfunctions could also be obtained by using raising and lowering operators throughout a factorization of the Hamiltonian $\Delta_{\alpha,\beta}$ in (2.7) based on the Lie algebra $su(1,1)$ commutation relations [10, pp. 3-4].

3 A COHERENT STATES FORMALISM

In this section, we adopt a new generalization of the canonical coherent states, which extend a well known generalization ([11, p. 4568]) by considering a kind of the identity resolution that we obtain as a limit with respect to a certain parameter. Precisely, we propose the following definition:

Definition 3.1. *Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{\psi_n\}_{n=0}^{+\infty}$. Let $\mathfrak{D} \subseteq \mathbb{C}$ be an open subset of $\mathbb{C}$ and let $\Phi_n : \mathfrak{D} \to \mathbb{C}, n = 0,1,2,\cdots$ be a sequence of complex functions. Define*

$$\mid z, \varepsilon > := (N_\varepsilon(z))^{-\frac{1}{2}}\sum_{n=0}^{+\infty}\frac{\Phi_n(z)}{\sqrt{\sigma_\varepsilon(n)}}\mid \psi_n >, \quad z \in \mathfrak{D}, \ \varepsilon > 0, \quad (3.1)$$

where $N_\varepsilon(z)$ is a normalization factor and $\sigma_\varepsilon(n), n = 0, 1, 2, \cdots$ a sequence of positive numbers depending on $\varepsilon > 0$. The set of vectors $\{| z, \varepsilon >, z \in \mathfrak{D}\}$ is said to form a set of generalized coherent states if :
(i) for each fixed $\varepsilon > 0$ and $z \in \mathfrak{D}$, the state $| z, \varepsilon >$ is normalized, that is $< z, \varepsilon \mid z, \varepsilon >_{\mathcal{H}} = 1$,
(ii) the states $\{| z, \varepsilon >, z \in \mathfrak{D}\}$ satisfy the following resolution of the identity

$$\lim_{\varepsilon \to 0^+} \int_{\mathfrak{D}} | z, \varepsilon >< z, \varepsilon | \, d\mu_\varepsilon(z) = \mathbf{1}_{\mathcal{H}} \tag{3.2}$$

where $d\mu_\varepsilon$ is an appropriately chosen measure and $\mathbf{1}_{\mathcal{H}}$ is the identity operator on the Hilbert space $\mathcal{H}$.

We should precise that, in the above definition, the Dirac's *bra-ket* notation $| z, \varepsilon >< z, \varepsilon |$ means the rank-one-operator $\varphi \longmapsto < \varphi \mid z, \varepsilon >_{\mathcal{H}}| z, \varepsilon >$, $\varphi \in \mathcal{H}$. Also, the limit in (ii) is to be understood as follows. Define the operator

$$\mathcal{O}_\varepsilon[\varphi](\cdot) := \left(\int_{\mathfrak{D}} | z, \varepsilon >< z, \varepsilon | \, d\mu_\varepsilon(z) \right) [\varphi](\cdot) \tag{3.3}$$

then the above limit (3.2) means that $\mathcal{O}_\varepsilon[\varphi](\cdot) \to \varphi(\cdot)$ as $\varepsilon \to 0^+$, *almost every where* with respect to $(\cdot)$.

Remark 3.1. The formula (3.1) can be considered as a generalization of the series expansion of the canonical coherent states

$$| z > := \left(e^{|z|^2} \right)^{-\frac{1}{2}} \sum_{n=0}^{+\infty} \frac{z^k}{\sqrt{n!}} | \phi_n >, z \in \mathbb{C} \tag{3.4}$$

with $\phi_n, n = 0, 1, 2, \cdots$ being an orthonormal basis in $L^2(\mathbb{R}, d\xi)$ of eigenstates of the harmonic oscillator, which is given by the functions $\phi_n(\xi) := \left(\sqrt{\pi} 2^n n! \right)^{-\frac{1}{2}} e^{-\frac{1}{2}\xi^2} H_n(\xi)$ where $H_n(\cdot)$ denotes nth Hermite polynomial ([8, p. 249]).

4 COHERENT STATES ATTACHED TO $\Delta_{\alpha,\beta}$

As announced in section 1, we now will construct a set of the normalized states labeled by points $x \in \mathbb{R}$ and depending on the parameters: $\theta \in]0, \pi[, \gamma > 1, \beta > 0$ and $\varepsilon > 0$. These states will be denoted $| x, \varepsilon >_{\theta,\gamma,\beta}$ and will belong to $L^2(\mathbb{R}_+, d\xi)$ the state Hilbert space of the Hamiltonian $\Delta_{\alpha,\beta}$ in (2.7).

Definition 4.1. *Define a set of states $\left(| x, \varepsilon >_{\theta,\gamma,\beta} \right)_{x \in \mathbb{R}}$ labeled by points $x \in \mathbb{R}$ and depending on the parameters $\theta \in]0, \pi[, \gamma > 1, \beta > 0$ and $\varepsilon > 0$ by*

$$| x, \varepsilon >_{\theta,\gamma,\beta} := \left(\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x) \right)^{-\frac{1}{2}} \sum_{m=0}^{+\infty} \frac{P_m^{\left(\frac{1}{2}\gamma\right)}(x, \theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(m)}} | \psi_m^{\gamma,\beta} > \tag{4.1}$$

with the precisions:
$\bullet \mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x)$ is a normalization factor such that $_{\theta,\gamma,\beta} < x, \varepsilon \mid x, \varepsilon >_{\theta,\gamma,\beta} = 1$

$\bullet P_m^{\left(\frac{1}{2}\gamma\right)}(x, \theta)$ are the Meixner-Pollaczek polynomials given by

$$P_m^{\left(\frac{1}{2}\gamma\right)}(x, \theta) = \frac{(\gamma)_m}{m!} e^{im\theta} {}_2F_1\left(-m, \frac{\gamma}{2} + ix, \gamma, 1 - e^{-2i\theta}\right) \tag{4.2}$$

where $_2F_1(.)$ denotes the Gauss hypergeometric function.
$\bullet \sigma_\varepsilon^{\beta,\gamma}(m), m = 0, 1, 2, \cdots$, are sequences of positive numbers given by

$$\sigma_\varepsilon^{\beta,\gamma}(m) := (m!)^{-1} (\gamma)_m e^{2\beta(2m+\gamma)\varepsilon}, \tag{4.3}$$

with $\gamma = \gamma(\alpha) = 1 + \frac{1}{2}\sqrt{1+4\alpha}$

● $| \psi_m^{\gamma,\beta} >, m = 0, 1, 2, \cdots$, *is the orthonormal basis of* $L^2(\mathbb{R}_+, d\xi)$ *given in* (2.6)

We shall give the main properties on these states in the following three propositions.

Proposition 4.2. *Let* $\theta \in]0, \pi[, \gamma > 1$ *and* $\varepsilon > 0$ *be fixed parameters. Then, the normalization factor in* (4.1) *has the following expression:*

$$\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x) = \frac{\left(1 - e^{-4\varepsilon\beta + 2i\theta}\right)^{2ix}}{(2sh2\varepsilon\beta)^{\gamma}} \times {}_2F_1\left(\frac{1}{2}\gamma + ix, \frac{1}{2}\gamma + ix, \gamma; \frac{-4e^{-4\varepsilon\beta}\sin^2\theta}{\left(1 - e^{-4\varepsilon\beta}\right)^2}\right) \qquad (4.4)$$

for every $x \in \mathbb{R}$.

Proof. calculate the normalization factor, we start by writing the condition

$$1 =_{\theta,\gamma,\beta} < x, \varepsilon \mid x, \varepsilon >_{\theta,\gamma,\beta} \qquad (4.5)$$

Eq.(4.5) is equivalent to

$$\left(\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x)\right)^{-1} \sum_{m=0}^{+\infty} \frac{1}{\sigma_\varepsilon^{\beta,\gamma}(m)} \left(P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)\right)^2 = 1, \qquad (4.6)$$

Inserting the expression (4.3) into Eq.(4.6), we obtain that

$$\mathcal{N}_{\theta,\gamma,\varepsilon}(x) = e^{-2\varepsilon\beta\gamma} \sum_{m=0}^{+\infty} \frac{m!}{(\gamma)_m} \left(e^{-4\varepsilon\beta}\right)^m \left(P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)\right)^2 \qquad (4.7)$$

Next, we make use of the following identity ([12, p. 527]):

$$\sum_{m=0}^{+\infty} \frac{m!}{(\gamma)_m} \mu^m P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta_1) P_m^{\left(\frac{1}{2}\gamma\right)}(y,\theta_2) = \left(1 - \mu e^{i(\theta_1 - \theta_2)}\right)^{-\frac{1}{2}\gamma - iy} \left(1 - \mu e^{i(\theta_2 - \theta_1)}\right)^{-\frac{1}{2}\gamma - ix} \qquad (4.8)$$

$$\times \left(1 - \mu e^{i(\theta_1 + \theta_2)}\right)^{ix+iy} {}_2F_1\left(\frac{1}{2}\gamma + ix, \frac{1}{2}\gamma + ix, \gamma; \frac{-4\mu \sin\theta_1 \sin\theta_2}{\left(1 - \mu e^{i(\theta_2 - \theta_1)}\right)\left(1 - \mu e^{i(\theta_1 - \theta_2)}\right)}\right)$$

for $\theta_1 = \theta_2 = \theta$, $x = y$ and $\mu = e^{-4\varepsilon\beta}$. Then, we arrive at the result

$$\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x) = \frac{\left(1 - e^{-4\varepsilon\beta + 2i\theta}\right)^{2ix}}{(2sh2\varepsilon\beta)^{\gamma}} \times {}_2F_1\left(\frac{1}{2}\gamma + ix, \frac{1}{2}\gamma + ix, \gamma; \frac{-4e^{-4\varepsilon\beta}(\sin\theta)^2}{\left(1 - e^{-4\varepsilon\beta}\right)^2}\right) \qquad (4.9)$$

This ends the proof.   □

Now, we will present a closed form for the constructed generalized coherent states as follows:

Proposition 4.3. *Let* $\theta \in]0, \pi[, \gamma > 1, \beta > 0$ *and* $\varepsilon > 0$ *be fixed parameters. Then, the wave functions of the states* $| x, \varepsilon >_{\theta,\gamma,\beta}$ *defined in* (4.1) *can be written in a closed form as*

$$< \xi \mid x, \varepsilon >_{\theta,\gamma,\beta} = \frac{\sqrt{2\beta\gamma}e^{-\varepsilon\beta\gamma}}{\sqrt{\Gamma(\gamma)}} \left|1 - e^{-2\beta\varepsilon + i\theta}\right|^{-\gamma} \left(\frac{1 - e^{-2\beta\varepsilon + i\theta}}{1 - e^{-2\beta\varepsilon - i\theta}}\right)^{ix} \left(\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x)\right)^{-\frac{1}{2}} \qquad (4.10)$$

$$\times \xi^{\gamma - \frac{1}{2}} \exp\left(-\frac{1}{2}\beta\xi^2 \left(\frac{1 + e^{-2\beta\varepsilon + i\theta}}{1 - e^{-2\beta\varepsilon + i\theta}}\right)\right) \times {}_1F_1\left(\frac{\gamma}{2} + ix, \gamma; \frac{2i\beta\xi^2 \sin\theta e^{-2\beta\varepsilon}}{\left|1 - e^{-2\beta\varepsilon + i\theta}\right|^2}\right)$$

for every $\xi \in \mathbb{R}_+$.

Proof. start by writing the expression of the wave function of states $\mid x, \varepsilon >_{\theta,\gamma,\beta}$ according to definition (4.1) as

$$< \xi \mid x, \varepsilon >_{\theta,\gamma,\beta} = \left(\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x) \right)^{-\frac{1}{2}} \sum_{m=0}^{+\infty} \frac{P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\theta,\gamma}(m)}} < \xi \mid \psi_m^{\gamma,\beta} >, \xi \in \mathbb{R}_+. \tag{4.11}$$

We have thus to look for a closed form of the series

$$\mathcal{S}(\xi) := \sum_{m=0}^{+\infty} \frac{P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(m)}} < \xi \mid \psi_m^{\gamma,\beta} > \tag{4.12}$$

which also reads

$$\mathcal{S}(\xi) = \sum_{m=0}^{+\infty} \frac{\sqrt{m!}}{\sqrt{(\gamma)_m}} e^{-\varepsilon\beta(2m+\gamma(\alpha))} P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta) < \xi \mid \psi_m^{\gamma,\beta} > \tag{4.13}$$

$$= e^{-\varepsilon\beta\gamma(\alpha)} \sum_{m=0}^{+\infty} \frac{\sqrt{m!}}{\sqrt{(\gamma)_m}} e^{-m2\beta\varepsilon} P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta) < \xi \mid \psi_m^{\gamma,\beta} > . \tag{4.14}$$

Replacing the Meixner-Pollaczek polynomial $P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)$ by its expression in terms of the Gauss hypergeometric function as given in (4.2), then Eq.(4.14) takes the form

$$\mathcal{S}(\xi) = e^{-\varepsilon\beta\gamma(\alpha)} \sum_{m=0}^{+\infty} \frac{\sqrt{(\gamma)_m}}{\sqrt{m!}} \left(e^{-2\beta\varepsilon+i\theta} \right)^m {}_2F_1(-m, \frac{\gamma}{2} + ix, \gamma, 1 - e^{-2i\theta}) < \xi \mid \psi_m^{\gamma,\beta} > \tag{4.15}$$

Put $\tau := e^{-2\beta\varepsilon+i\theta}, |\tau| = e^{-2\beta\varepsilon} < 1$. Then Eq.(4.15) becomes

$$\mathcal{S}(\xi) = e^{-\varepsilon\beta\gamma(\alpha)} \sum_{m=0}^{+\infty} \frac{\sqrt{(\gamma)_m}}{\sqrt{m!}} \tau^m {}_2F_1(-m, \frac{\gamma}{2} + ix, \gamma, 1 - e^{-2i\theta}) < \xi \mid \psi_m^{\gamma,\beta} > . \tag{4.16}$$

Replacing the wavefunction $< \xi \mid \psi_m^{\gamma,\beta} >$ by its expression in (2.9), we get that

$$\mathcal{S}(\xi) = e^{-\varepsilon\beta\gamma(\alpha)} \sum_{m=0}^{+\infty} \frac{\sqrt{(\gamma)_m}}{\sqrt{m!}} \tau^m {}_2F_1(-m, \frac{\gamma}{2} + ix, \gamma, 1 - e^{-2i\theta}) \tag{4.17}$$

$$\times \left(\frac{2\beta^\gamma m!}{\Gamma(\gamma+m)} \right)^{\frac{1}{2}} \xi^{\gamma-\frac{1}{2}} \exp\left(-\frac{1}{2}\beta\xi^2 \right) L_m^{(\gamma-1)}(\beta\xi^2)$$

Now, we summarize up the above calculations by writing

$$< \xi \mid x, \varepsilon >_{\theta,\gamma,\beta} = \frac{\sqrt{2\beta^\gamma}}{\sqrt{\Gamma(\gamma)}} e^{-\varepsilon\beta\gamma} \left(\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x) \right)^{-\frac{1}{2}} \xi^{\gamma-\frac{1}{2}} e^{-\frac{1}{2}\beta\xi^2} \mathfrak{S}_{\gamma,\beta}^{\tau,x,\theta}(\xi) \tag{4.18}$$

where

$$\mathfrak{S}_{\gamma,\beta}^{\tau,x,\theta}(\xi) := \sum_{m=0}^{+\infty} \tau^m {}_2F_1(-m, \frac{\gamma}{2} + ix, \gamma, 1 - e^{-2i\theta}) L_m^{(\gamma-1)}(\beta\xi^2) \tag{4.19}$$

Next, with the help of the generating formula [13, p. 213]:

$$\sum_{n=0}^{+\infty} t^n {}_2F_1(-n, c, 1+v; y) L_n^{(v)}(u) = (1-t)^{-1+c-v} (1-t+yt)^{-c} \tag{4.20}$$

$$\times \exp\left(\frac{-ut}{1-t} \right) {}_1F_1\left(c, 1+v, \frac{yut}{(1-t)(1-t+yt)} \right)$$

for $t = \tau, n = m, c = \frac{\gamma}{2} + ix, y = 1 - e^{-2i\theta}, v = \gamma - 1$ and $u = \beta\xi^2$, we obtain an expression of series (4.19) as

$$\mathfrak{S}_{\gamma,\beta}^{\tau,x,\theta}(\xi) = (1 - \tau)^{-\frac{\gamma}{2} + ix} \left(1 - e^{-2i\theta}\tau\right)^{-\left(\frac{\gamma}{2} + ix\right)} \exp\left(-\frac{\beta\zeta^2\tau}{1 - \tau}\right) \qquad (4.21)$$

$$\times \exp(-\frac{\beta\tilde{\zeta}^2\tau}{1 - \tau})_1F_1\left(\frac{\gamma}{2} + ix, \gamma, \frac{(1 - e^{-2i\theta})\beta\tilde{\zeta}^2\tau}{(1 - \tau)(1 - e^{-2i\theta}\tau)}\right)$$

Finally, we arrive at the following expression of the wave functions

$$< \xi \mid x, \varepsilon >_{\theta,\gamma,\beta} = \frac{\sqrt{2\beta^\gamma}e^{-\varepsilon\beta\gamma}}{\sqrt{\Gamma(\gamma)}} \left|1 - e^{-2\beta\varepsilon + i\theta}\right|^{-\gamma} \left(\frac{1 - e^{-2\beta\varepsilon + i\theta}}{1 - e^{-2\beta\varepsilon - i\theta}}\right)^{ix} (\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x))^{-\frac{1}{2}} \qquad (4.22)$$

$$\times \zeta^{\gamma - \frac{1}{2}} \exp\left(-\frac{1}{2}\beta\zeta^2 \left(\frac{1 + e^{-2\beta\zeta + i\theta}}{1 - e^{-2\beta\zeta + i\theta}}\right)\right) \times_1 F_1\left(\frac{\gamma}{2} + ix, \gamma; \frac{2i\beta\zeta^2 \sin\theta e^{-2\beta\varepsilon}}{\left|1 - e^{-2\beta\varepsilon + i\theta}\right|^2}\right)$$

This ends the proof.   □

Proposition 4.4. *The states $\mid x, \varepsilon > \equiv \mid x, \varepsilon >_{\theta,\gamma,\beta}, x \in \mathbb{R}$ satisfy the following resolution of the identity*

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \mid x, \varepsilon >< \varepsilon, x \mid d\mu_{\theta,\gamma,\varepsilon}(x) = \mathbf{1}_{L^2(\mathbb{R}_+, d\xi)} \qquad (4.23)$$

where $\mathbf{1}_{L^2(\mathbb{R}_+, d\xi)}$ is the identity operator on the Hilbert space $L^2(\mathbb{R}_+, d\xi)$ and $d\mu_{\theta,\gamma,\varepsilon}(x)$ is a measure on $\mathbb{R}$ with the expression

$$d\mu_{\theta,\gamma,\beta,\varepsilon}(x) := \frac{(2\sin\theta)^{\gamma-1}}{\pi^{\gamma-1}\Gamma(\gamma)\cos ec(\theta)}\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x)e^{-(\pi-2\theta)x} \left|\Gamma\left(\frac{\gamma}{2} + ix\right)\right|^2 dx, \qquad (4.24)$$

$\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x)$ being the normalization factor given in proposition (4.1).

Proof. Let us assume that the measure takes the form

$$d\mu_{\theta,\gamma,\beta,\varepsilon}(x) = \mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x)Y_{\theta,\gamma}(x)dx \qquad (4.25)$$

where $Y_{\theta,\gamma}(x)$ is an auxiliary density to be determined. Let $\varphi \in L^2(\mathbb{R}_+, d\xi)$ and let us start by writing the following action

$$\mathcal{O}_{\theta,\gamma,\beta,\varepsilon}[\varphi] := \left(\int_{\mathbb{R}} \mid x, \varepsilon >< x, \varepsilon \mid d\mu_{\theta,\gamma,\beta,\varepsilon}(x)\right)[\varphi] \qquad (4.26)$$

$$= \int_{\mathbb{R}} < \varphi \mid x, \varepsilon >< x, \varepsilon \mid d\mu_{\theta,\gamma,\beta,\varepsilon}(x) \qquad (4.27)$$

We make use of the definition of $\mid x, \varepsilon >$ given in (4.1) :

$$\mathcal{O}_{\theta,\gamma,\beta,\varepsilon}[\varphi] = \int_{\mathbb{R}} \left\langle \varphi, (\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x))^{-\frac{1}{2}} \sum_{m=0}^{+\infty} \frac{P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(m)}} \Big| \psi_m^{\gamma,\beta} \right\rangle < x, \varepsilon \mid d\mu_{\theta,\gamma,\beta,\varepsilon}(x) \tag{4.28}$$

$$= \int_{\mathbb{R}} \sum_{m=0}^{+\infty} \frac{P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(m)}} < \varphi \mid \psi_m^{\gamma,\beta} > < x, \varepsilon \mid (\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x))^{-\frac{1}{2}} d\mu_{\theta,\gamma,\beta,\varepsilon}(x) \tag{4.29}$$

$$= \left(\sum_{m,j=0}^{+\infty} \int_{\mathbb{R}} \frac{P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(m)}} \frac{P_j^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(j)}} \mid \psi_m^{\gamma,\beta} > < \psi_j^{\gamma,\beta} \mid (\mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x))^{-1} d\mu_{\theta,\gamma,\beta,\varepsilon}(x) \right) [\varphi] \tag{4.30}$$

Replace $d\mu_{\theta,\gamma,\beta,\varepsilon}(x) = \mathcal{N}_{\theta,\gamma,\beta,\varepsilon}(x) Y_{\theta,\gamma}(x)dx$, then Eq. (4.30) takes the form

$$\mathcal{O}_{\theta,\gamma,\beta,\varepsilon} = \sum_{m,j=0}^{+\infty} \left[\int_{\mathbb{R}} \frac{P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(m)}} \frac{P_j^{\left(\frac{1}{2}\gamma\right)}(x,\theta)}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(j)}} Y_{\theta,\gamma}(x)dx \right] \mid \psi_m^{\gamma,\beta} > < \psi_j^{\gamma,\beta} \mid \tag{4.31}$$

Then, we need to consider the integral

$$I_{m,j}(\theta,\gamma,\varepsilon) := \int_{\mathbb{R}} P_m^{\left(\frac{1}{2}\gamma\right)}(x,\theta) P_j^{\left(\frac{1}{2}\gamma\right)}(x,\theta) Y_{\theta,\gamma}(x)dx \tag{4.32}$$

We recall the orthogonality relations of the Meixner-Pollaczek polynomials ([14, p. 764]):

$$\int_{\mathbb{R}} P_m^{(\nu)}(x,\theta) P_j^{(\nu)}(x,\theta) \omega_\nu(x,\theta) \, dx = \frac{\Gamma(2\nu+m)\cos ec\theta}{m!} \delta_{m,j} \tag{4.33}$$

where

$$\omega_\nu(x,\theta) = \frac{(2\sin\theta)^{2\nu-1}}{\pi \cos ec(\theta)} e^{-(\pi-2\theta)x} \left| \Gamma(\nu+ix) \right|^2. \tag{4.34}$$

This suggests us to set

$$Y_{\theta,\gamma}(x) := \frac{\omega_{\frac{1}{2}\gamma}(x,\theta)}{\Gamma(\gamma)\cos ec(\theta)}. \tag{4.35}$$

Therefore, (4.32) reduces to

$$I_{m,j}(\theta,\gamma,\varepsilon) = \frac{(\gamma)_m}{m!} \delta_{m,j} \tag{4.36}$$

which means that the operator in (4.31) takes the form:

$$\mathcal{O}_{\theta,\gamma,\beta,\varepsilon} \equiv \mathcal{O}_{\gamma,\beta,\varepsilon} = \sum_{m,j=0}^{+\infty} \frac{1}{\sqrt{\sigma_\varepsilon^{\beta,\gamma}(m)}\sqrt{\sigma_\varepsilon^{\beta,\gamma}(j)}} \frac{(\gamma)_m}{m!} \delta_{m,j} \mid \psi_m^{\gamma,\beta} > < \psi_j^{\gamma,\beta} \mid \tag{4.37}$$

$$= \sum_{m=0}^{+\infty} \frac{(\gamma)_m}{m!} \frac{1}{\sigma_\varepsilon^{\beta,\gamma}(m)} \mid \psi_m^{\gamma,\beta} > < \psi_m^{\gamma,\beta} \mid \tag{4.38}$$

Recalling the expression

$$\sigma_\varepsilon^{\beta,\gamma}(m) := (m!)^{-1}(\gamma)_m e^{2\beta(2m+\gamma)\varepsilon}, \tag{4.39}$$

we arrive at

$$\mathcal{O}_{\theta,\gamma,\varepsilon}[\varphi] = \sum_{m=0}^{+\infty} e^{-2\beta(2m+\gamma)\varepsilon} \left(\mid \psi_m^{\gamma,\beta} > < \psi_m^{\gamma,\beta} \mid \right) [\varphi]. \tag{4.40}$$

For $u \in \mathbb{R}_+$, we can write

$$\mathcal{O}_{\theta,\gamma,\varepsilon}[\varphi](u) = \sum_{m=0}^{+\infty} e^{-2\beta(2m+\gamma)\varepsilon} < \varphi \mid \psi_m^{\gamma,\beta} > \psi_m^{\gamma,\beta}(u) \tag{4.41}$$

$$= \sum_{m=0}^{+\infty} e^{-2\beta(2m+\gamma)\varepsilon} \left(\int_0^{+\infty} \varphi(\xi) \psi_m^{\gamma,\beta}(\xi) d\xi \right) \psi_m^{\gamma,\beta}(u) \tag{4.42}$$

$$= \int_0^{+\infty} \varphi(\xi) \left(\sum_{m=0}^{+\infty} e^{-2\beta(2m+\gamma)\varepsilon} \psi_m^{\gamma,\beta}(\xi) \psi_m^{\gamma,\beta}(u) \right) d\xi \tag{4.43}$$

We are then lead to calculate the sum

$$\mathcal{G}_\varepsilon^{\alpha,\beta}(u,\xi) := \sum_{m=0}^{+\infty} e^{-2\beta(2m+\gamma)\varepsilon} \psi_m^{\gamma,\beta}(\xi) \psi_m^{\gamma,\beta}(u). \tag{4.44}$$

For this we recall the explicit expression of the wavefunction $\psi_m^{\gamma,\beta}(\xi)$ in (2.9). So that the above sum reads

$$\mathcal{G}_\varepsilon^{\alpha,\beta}(u,\xi) = e^{-2\beta\gamma\varepsilon} 2\beta^\gamma (\xi u)^{\gamma-\frac{1}{2}} e^{-\frac{1}{2}\beta(u^2+\xi^2)}$$

$$\times \sum_{m=0}^{+\infty} \mu^m \frac{m!}{\Gamma(m+\gamma)} L_m^{(\gamma-1)}(\beta u^2) L_m^{(\gamma-1)}(\beta\xi^2). \tag{4.45}$$

Eq. (4.45) can be rewritten as

$$\mathcal{G}_\varepsilon^{\alpha,\beta}(u,\xi) = e^{-2\beta\gamma\varepsilon} 2\beta^\gamma (u\xi)^{\gamma-\frac{1}{2}} e^{-\frac{1}{2}\beta(u^2+\xi^2)} K\left(e^{-4\beta\varepsilon}; u,\xi\right) \tag{4.46}$$

where we have introduced the kernel function

$$K(\rho; u,\xi) := \sum_{m=0}^{+\infty} \rho^m \frac{m!}{\Gamma(m+\gamma)} L_m^{(\gamma-1)}(\beta u^2) L_m^{(\gamma-1)}(\beta\xi^2), 0 < \rho < 1. \tag{4.47}$$

Returning back to Eq. (4.43) and taking into account Eq. (4.47, we get that

$$\mathcal{O}_{\theta,\gamma,\varepsilon}[\varphi](u) = 2\beta^\gamma e^{-2\beta\varepsilon} u^{\gamma-1} e^{-\frac{1}{2}\beta u^2} \int_0^{+\infty} \xi^{\gamma-\frac{1}{2}} e^{-\frac{1}{2}\beta\xi^2} K\left(e^{-4\beta t}, \beta u^2, \beta\xi^2\right) \varphi(\xi) d\xi \tag{4.48}$$

We split the right hand side of Eq. (4.48)

$$\mathcal{O}_{\theta,\gamma,\varepsilon}[\varphi](u) = \vartheta_{\beta,\gamma,\varepsilon}(u) M[\varphi](u), \tag{4.49}$$

where

$$M[\varphi](u) = \frac{1}{2}\beta^{-\frac{1}{2}\gamma-\frac{1}{4}} \int_0^{+\infty} K(\rho; w,s) h(s) s^{\gamma-1} e^{-s} ds, \tag{4.50}$$

with $w := \beta u^2$, and

$$h(s) := s^{-\frac{1}{2}\gamma+\frac{1}{4}} e^{\frac{1}{2}s} \varphi\left(\beta^{-\frac{1}{2}}\sqrt{s}\right). \tag{4.51}$$

By direct calculations, one can check that $h \in L^2\left(\mathbb{R}_+, s^{\gamma-1} e^{-s} ds\right)$. Precisely, we have that

$$\|h\|_{L^2(\mathbb{R}_+, s^{\gamma-1} e^{-s} ds)}^2 = 2\sqrt{\beta} \|\varphi\|_{L^2(\mathbb{R}_+)} \tag{4.52}$$

We are now in position to apply the result of B. Muckenhoopt [15] who considered the Poisson integral of a function $f \in L^p\left(\mathbb{R}^+, s^\eta e^{-s}ds\right), \eta > -1, 1 \leq p \leq +\infty$ defined by

$$A\left[f\right]\left(\rho, w\right) := \int_0^{+\infty} K\left(\rho, w, s\right) f\left(s\right) s^\eta e^{-s}ds, 0 < \rho < 1 \qquad (4.53)$$

with the kernel $K\left(\rho, w, s\right)$ defined as in (4.47). He proved that $\lim_{\rho \to 1^-} A\left[f\right]\left(\rho, x\right) = f(x)$ almost everywhere in $[0, +\infty[, 1 \leq p \leq \infty$. We apply this result in the case $p = 2, f = h$ and $A \equiv M$ to obtain that

$$M[\varphi](u) \to \frac{1}{2}\beta^{-\frac{1}{2}\gamma - \frac{1}{4}}h(\beta u^2) = \beta^{-\frac{1}{2}\gamma + \frac{1}{4}}u^{-\gamma + \frac{1}{2}}e^{\frac{1}{2}\beta u^2}\varphi(u) \qquad (4.54)$$

Recalling that $\rho = e^{-4\beta t}$, we get that

$$\mathcal{O}_{\theta,\gamma,\varepsilon}[\varphi]\left(u\right) = \vartheta_{\beta,\gamma,\varepsilon}(u)M\left[\varphi\right]\left(u\right) \to \varphi(u) \text{ as } \varepsilon \to 0^+ \qquad (4.55)$$

which means that

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \mid x, \varepsilon >< x, \varepsilon \mid d\mu_{\theta,\gamma,\varepsilon}(x) = \mathbf{1}_{L^2(\mathbb{R}_+, d\xi)}. \qquad (4.56)$$

This ends the proof. □

5 SUMMARY

We have been a concerned with the model of the Gol'dman-Krivchenkov Hamiltonian acting on the Hilbert space of square integrable functions on the positive real half-line. We have constructed a family of coherent states labeled by points of the whole real line, depending on some parameters and belonging to the state Hilbert space of this Hamiltonian. We have adopted a generalized formalism of canonical coherent states when written as superpositions of the harmonic oscillator number states. That is, we have presented our generalized coherent states as a superposition of an orthonormal basis of the state Hilbert space of the Hamiltonian. The Meixner-Pollaczek polynomials have been chosen to play the role of coefficients in this superposition. This choice enables us to present the constructed states in a closed form. The state Hilbert space identity is solved by an new way as a limit with respect to a certain parameter by exploiting the results of B. Muckenhoupt on Poisson integrals for Laguerre expansions. The obtained result will lead to investigate the class of special functions and orthogonal polynomials that could play the role of *coefficients* in the series expansion of coherent states together with the class of functions that could be considered as *number states* for a certain Hamiltonian in a such way that the present procedure works.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCES AND TECHNICS (M'GHILA), PO. BOX 523, SULTAN MY SLIMANE UNIVERSITY, CP. 23 000, BÉNI MELLAL, MOROCCO

E-mail address: mouayn@fstbm.ac.ma