Recall:

- (X,T) is a linear dynamical system if T is linear and cts and $T: X \to X$ with X Fréchet.

T is <u>hypercyclic</u> if $\exists x \in X$ s.t $orb(x,T) = \{x, Tx, T^2x \dots\}$ is dense in X. $\Leftrightarrow$ T is topologically transitive

T is <u>weakly mixing</u> if $T \oplus T$ is hypercyclic on X^2.

T is <u>mixing</u> if for any U & V, non-empty open, $\exists N \in \mathbb{N}$ s.t $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

<u>Hypercyclicity criterion</u>: If there exists dense $X_0, Y_0 \subset X$ a sequence of increasing positive integers $(n_k)_k$ and $S_k : Y_0 \to X$ s.t

$$\left.\begin{array}{lll} (i) & T^{n_k} \to 0 & \text{ptw on } X_0 \\ (ii) & S_k \to 0 & \text{ptw on } Y_0 \\ (iii) & T^{n_k} S_k \to Id & \text{ptw on } Y_0 \end{array}\right\} \text{ then } T \text{ is hypercyclic.}$$

Mixing $\Rightarrow$ Weakly mixing $\Rightarrow$ hypercyclicity

$\not\Leftarrow$ not always. $\not\Leftarrow$ not always.

Weakly mixing $\Updownarrow$ Hypercyclicity Criterion.

If T satisfies HC then $T \oplus T$ also does. (Use $X_0 \oplus X_0$, $Y_0 \oplus Y_0$, $S_k \oplus S_k$, n_k for the proof).

<u>Theorem</u> (Bès, Peris 1999): T is weakly mixing $\Leftrightarrow$ T satisfies HC

<u>Proof</u>: Assume $T \oplus T$ is hypercyclic, (x,y) is a hypercyclic vector for $T \oplus T$. and $X_0 = Y_0 = orb(x,T)$. T is hypercyclic $\Rightarrow$ T has a dense range $\Rightarrow$ T^N has dense range. $(x, T^N y)$ is hypercyclic vector for $T \oplus T$

$\{(T^n x, T^n(T^N y)) : n \geq 0\} = (I \oplus T^N)(\{(T^n x, T^n y)\}, n \geq 0)$ is dense in $X \times X$. Then for any neighborhood U of zero $\exists (x,u)$ s.t $u \in U$, it's hypercyclic vector for $T \oplus T$.

Choose $(u_k) \subseteq X^{\mathbb{N}}$, $(n_k) \subset \mathbb{N}^{\mathbb{N}}$ s.t (Notation $\|v\| := dist(v,0)$ for $v \in X$).

$\|u_k\| < \frac{1}{k}$ $\|T^{n_k} x\| < \frac{1}{k}$ and $\|T^{n_k} u_k - x\| < \frac{1}{k}$

- $T^{n_k} \to 0$ on $X_0 = orb(x,T)$

Define $S_k : Y_0 \to X$ s.t $S_k(T^n x) = T^n u_k$ well-defined.

$S_k(T^n x) = T^n u_k \to 0$ so $S_k \to 0$ on $Y_0 = orb(x,T)$.

$T^{n_k} S_k(T^n x) = T^{n_k} T^n u_k = T^n(T^{n_k} u_k) \to T^n x$ so $T^{n_k} S_k \to Id$ on $Y_0 = orb(x,T)$.

<u>Ex</u> $c_0 = \{(x_n) : x_n \to 0\}$ with sup norm $\sup_n |x_n|$

$B_w(x_1, x_2, \dots) = (w_2 x_2, w_3 x_3, \dots)$ B is cts if $\sup_n |w_n| < \infty$. $w = (w_1, w_2, \dots)$

Let $w = (1, 2, \underset{m_1}{2^{-1}}, 2, 2, 2^{-1}, \underset{m_2}{2^{-1}}, 2, 2, 2, 2^{-1}, 2^{-1}, \underset{m_3}{2^{-1}}, \dots)$

Choose m_k at $w_{m_k} = 2^{-1}$ and $w_{m_k+1} = 2$

$B_w^n x = ((\prod_{j=2}^{n+1} w_j) x_{n+1}, \dots)$ $B_w^{m_k - 1} x = ((\prod_{j=1}^{m_k} w_j) x_{n+1}, \dots) = (x_{n+1}, \dots)$

Choose $V = \{x \in c_0, |x_1| > 1\}$ and $U = \{x \in c_0, \|x\| < 1\}$.

$B^{m_k-1}(U) \cap V = \emptyset$ So B_ω is not mixing

Let X_0, Y_0 = be the set of finite sequences ($X_0 = \text{span}\{e_n\}$)

$B_\omega^n \to 0$ on X_0

Define $Sx := (0, \omega_2^{-1}x, \omega_3^{-1}x_2, \ldots)$ cont. lin.

Define $n_k = m_k + k - 1$ for $k \geq 1$.

$$S^{n_k}e_1 = (0, 0, \ldots, 0, \prod_{j=2}^{m_k+k} \omega_j^{-1}, 0, \ldots) = (0, 0, 0, \ldots, 0, 2^{-k}, 0, \ldots 0) \to 0.$$

$e_1 = (1, 0, \ldots)$

$B_\omega S = Id.$

Hence weakly mixing $\not\Rightarrow$ mixing.

$X: \ell^p$ or c_0 $(1 \leq p < \infty)$, B_ω associated with $\omega = (\omega_1, \omega_2, \ldots)$ is

hypercyclic $\Leftrightarrow$ weakly mixing $\Leftrightarrow$ $\sup_n |\prod_{j=1}^{n} \omega_j| = \infty$.

- mixing $\Leftrightarrow$ $\lim_n |\prod_{j=1}^{n} \omega_j| = \infty$.

$T^n(U) \cap V \neq \emptyset \Leftrightarrow U \cap T^{-n}(V) \neq \emptyset$. (if T is invertible).

X, Fréchet space, X^* the space of continuous functionals $x^*: X \to \mathbb{C}$ define seminorms $p_{x^*}(x) = |\langle x, x^*\rangle| = |x^*(x)|$ $p_{x^*}: X \to \mathbb{R}_+$

Using these seminorms define weak topology on X

$x_n \xrightarrow{w} x \Leftrightarrow p(x_n - x) \to 0$ for all seminorms.

T is weakly hypercyclic if $orb(x, T)$ weakly dense in X for some x.

$\exists T \in \ell^2$ s.t T is weakly hypercyclic but T^{-1*} is not weakly hypercyclic.

$\exists T$ a weak hypercyclic operator but not norm hypercyclic.

<u>Question</u>: If T is weak hypercyclic, does it follow that the set of weakly hypercyclic vectors is a dense G_δ set?

<u>Question</u>: Does every Banach space X support a weakly hypercyclic vector that is not norm hypercyclic?

Somewhere Dense Orbits:

[Ansari]: If T is hypercyclic, then so is T^n for any $n \in \mathbb{N}$.

Assume x is a hypercyclic vector for T.

$orb(x, T) = orb(x, T^n) \cup orb(Tx, T^n) \cup \ldots \cup orb(T^{n-1}x, T^n)$

(Costakis-Peris) $\bigcup_{j=1}^{n} orb(x_j, T)$ is dense in $X \Rightarrow T$ is hypercyclic.

$T: X \to X$ | $D(x) = \overline{orb(x, T)}$ $U(x) = \text{int}\, D(x)$

(i) if $y \in D(x)$, then $D(y) \subset D(x)$. [follows from cty of T]

(ii) $U(x) = U(T^n x)$ [Excluding finite members from orbit does not effect the denseness of orbit]

(iii) If $R: X \to X$ cts s.t $RT = TR$ then $R(D(x)) \subset D(R(x))$

Lemma 1: If T admits a somewhere dense orbit and $p \neq 0$ is a polynomial then $p(T)$ has a dense range.

Proof: Claim: T^* can't have an eigenvalue: suppose not then

$$\langle T^n x, x^* \rangle = \langle x, (T^*)^n x^* \rangle = \lambda^n \underbrace{\langle x, x^* \rangle}_{\text{this can not be dense.}} ※$$

Claim: $T - \lambda I$ has a dense range for any $\lambda \in \mathbb{C}$.

$0 = \langle (T - \lambda I)x, x^* \rangle = \langle x, T^* x^* - \lambda x^* \rangle \Rightarrow$ if $T^* x^* - \lambda x^* = 0 \Rightarrow x^* = 0$ since T^* can't have an eigenvalue.

Lemma 2: If $orb(x,T)$ is somewhere dense then $A = \{ p(T)x : p \neq$ polynomial$\}$ is connected and dense.

Proof: A path connected: p, q s.t. $p \neq \lambda q$ then $t \to t p(T)x + (1-t) q(T) x$ $t \in [0,1]$. If $p = \lambda q$ choose r s.t. $r \neq \lambda p$, $\neq \lambda_2 q$

$\bar{A} \supset \overline{orb(x,T)}$ and $\bar{A}$ a subspace. $\exists x_0 \in X$ and 0-neighborhood W s.t. $x_0 + W \subset \bar{A}$. Let $y \in X$, $y \in \lambda W$ and $y \in \lambda(x_0 + W - x_0) = \lambda W$, $y \in \bar{A}$.

Theorem: (Bourdon, Feldman) If $orb(x,T)$ is somewhere dense in X then it's dense in X.

Proof: We want to show $U(x) \neq \emptyset$ then $D(x) = X$.

Step 1: We have that $T(X \setminus U(x)) \subset X \setminus U(x)$

Equivalently show $T^{-1}(U(x)) \subset U(x)$

- $y \in T^{-1}(U(x))$ and let V be a neighborhood of y.
- $U(x) \neq \emptyset \Rightarrow x_m := T^m x \in U(x)$
- by property (ii) $U(x_m) = U(T^m x) = U(x) \Rightarrow x_m$ has a somewhere dense orbit.

By lemma 2, $\exists p \neq 0$ s.t $p(T)x_m \subset V \cap T^{-1}(U(x))$ and by property (ii)

$p(T)x_m \in P(T)(U(x)) = p(T)(U(T^m x)) \subset p(T)(D(T^{m+1}x))$

$Tp(T)x_m \subset U(x) \subset D(x)$, by Property (i) and (iii).

$p(T)\ (D(T^{m+1}x)) \underset{(iii)}{\subset} D(Tp(T)x_m) \underset{(i)}{\subset} D(x)$

Therefore $V \cap D(x) \neq \emptyset$. $\Rightarrow y \in D(x)$ since $D(x)$ is a closed set.

$\Rightarrow T^{-1}(U(x)) \subset D(x)$ and from cty of T, T^{-1} of an open set is open so $T^{-1}(U(x)) \subset U(x)$ ↳ this is the biggest open set in $D(x)$.

Step 2: For any $z \in X \setminus U(x)$, $D(z) \subset X \setminus U(x)$. By definition of $D(x)$, we have the result.

$X \setminus U(x)$ is T-invariant and closed and T is cts.

Step 3: For any $p \neq 0$, $p(T)x \in X \setminus \partial D(x)$: $\hookrightarrow$ Boundary of $D(x)$.

- $\exists p \neq 0$ s.t $P(T)x \in \partial D(x)$
- $\exists y \in X$ s.t $p(T)y \in U(x)$
- $P(T)x \notin U(x)$ by property (iii) and step 2

$$P(T)(D(x)) \subset D(\underbrace{P(T)x}_{\notin U(x)}) \subset X \setminus U(x) \Rightarrow y \in X \setminus D(x).$$

By lemma 2, choose $q \neq 0$ s.t $q(T)x$ is very close to y thus $P(T)q(T)x \in U(x)$

- Since $P(T)x \in D(x)$, property (iii) and step 2 we have

$$P(T)q(T)x = q(T)p(T)x \subset q(T)(D(x)) \subset D(q(T)x) \subset X \setminus U(x). \text{ ※ So}$$

we can not have $p(T)x \in \partial D(x)$.

Step 4: $D(x) = X$

$A = \{p(T)x,\ p \neq 0, \text{polynomial}\} \subset \underbrace{X \setminus D(x) \cup U(x)}_{\text{disjoint}}$. A is dense $A \cap U(x) \neq \emptyset$

so $A \subset U(x) \Rightarrow A \subset D(x) \xRightarrow{A \text{ is dense}} D(x) = X$.

FREQUENT HYPERCYCLICITY!

Let μ be a probability measure on X, X Fréchet space. Defined on a Borel σ-algebra, that is, smallest σ-algebra containing open sets.

- $T: X \to X$ continuous and $\mu(T^{-1}(A)) = \mu(A)$ for $A \in \mathcal{B}(X)$ $\hookrightarrow$ Borel σ-algebra.
- T is ergodic if for any two $A, B \in \mathcal{B}(X)$ with $\mu(A) > 0$, $\mu(B) > 0$ then $\exists n \in \mathbb{N}$ s.t $\mu(T^n(A) \cap B) > 0$.
- If $\mu(U) > 0$ for any open U then ergodicity $\Rightarrow$ topologic transitivity.
  $\hookrightarrow$ (μ is of full support)

Birkoff Ergodicity Theorem: If T is ergodic w.r.t μ and f is a μ-measurable function on X, then

$$\frac{1}{N+1} \sum_{n=0}^{N} f(T^n(x)) \to \int_X f \, d\mu \quad \text{for } \mu\text{-almost all } x.$$

- X is separable then $\exists$ countable base $(U_k)_k$ for the topology of X. Apply B.E.T. to 1_{U_k} then

left hand side: $\displaystyle \frac{1}{N+1} \sum_{n=0}^{N} 1_{U_k}(T^n x) = \frac{\text{card}\{0 \leq n \leq N : T^n x \in U_k\}}{N+1}$

right hand side $\displaystyle \int_X 1_{U_k} \, d\mu = \mu(U_k) > 0$ (μ is of full support)

There are subsets $A_k \subset X$, $k \geq 1$, of full measure s.t $\forall x \in A_k$

$$\lim_{N \to \infty} \frac{\text{card}\{0 \leq n \leq N, T^n x \in U_k\}}{N+1} > 0$$

- Any open set includes a U_k and $\bigcap_k A_k$ is of full measure

$$\liminf_N \frac{\operatorname{card}\{0 \le n \le N : T^n x \in U\}}{N+1} > 0$$

<u>Definition</u>: $A \subseteq \mathbb{N}$, then $\operatorname{dens}(A) = \liminf_N \frac{\operatorname{card}\{n \le N : n \in A\}}{N+1}$

<u>Definition</u>: $T : X \to X$ is <u>frequently hypercyclic</u> If $\exists x \in X$ s.t for any non-empty open $U \subset X$
$\operatorname{dens}\{n : T^n x \in U\} > 0$.

chaotic $\Longrightarrow$
mixing $\Longrightarrow$ weakly mixing $\Longrightarrow$ hypercyclic $\Longrightarrow$ weakly hypercyclic
freq. hyper $\Longrightarrow$
freq hypercyclic $\not\Rightarrow$ chaotic, mixing
mixing $\not\Rightarrow$ freq. hypercyclic.

<u>Open Question</u>: Chaoticity $\overset{?}{\Longrightarrow}$ Frequently hypercyclicity.

<u>Definition</u>: $B \subset \mathbb{N}$ is syndetic if it has bounded gap: i.e. its complement does not contain intervals of arbitrary length.

<u>Theorem</u>: (Erdős, Sárközy). Let $A \subset \mathbb{N}$ be a set of positive lower density. Then $A - A = \{m - n : m, n \in A,\ m \ge n\}$ is syndetic.

<u>Theorem</u>: Frequently hypercyclic $\Longrightarrow$ weakly mixing.

<u>Proof</u>: $N(A, B) := \{n \in \mathbb{N} : T^n(A) \cap B \ne \emptyset\}$. Want to show for non-empty open $U, V, W \subset X$ with $0 \in W$ we have $N(U, W) \cap N(W, V) \ne \emptyset$ (Blow up/Collapse $\Updownarrow$ weakly mixing)

- T hypercyclic $\Rightarrow \exists\, n_0$ s.t $T^{n_0}(U) \cap W \ne \emptyset$ and $\exists\, U_0 \subset U$ s.t $T^{n_0}(U_0) \subset W$
- $x \in FHC(T)$ then $\exists A \subseteq \mathbb{N}$ s.t $T^n x \in U_0$ for $n \in A$ with $\operatorname{dens}(A) > 0$.
- $m, n \in A$, $m \ge n$ $\quad T^{n_0 + m - n}(T^n x) = T^{n_0 + m}(x) = T^{n_0}(T^m x) \in W$.

$n_0 + m - n \in n_0 + \underbrace{(A - A)}_{\text{syndetic}} \subset N(U_0, W)$

$\Downarrow$

$N(U_0, W)$ is syndetic.

T cts and linear $\Rightarrow T^{-k}(W)$ are 0-neighborhoods, for any $m \ge 1$ $\exists$ 0-nghd W_0 s.t $T^k(W_0) \subset W$ for $k = 1, \dots, m$.
By topological transitivity $\exists K > m$ and some $y \in N_0$ s.t $T^K y \in V$.
Therefore, for all $1 \le k \le M$

$$T^{K-k}(T^k y) \in T^{K-k}(W) \cap V$$ so for any $m > 1$

$N(W, V)$ contains m consecutive integers, from $N(W, V)$ being syndetic
$N(U, W) \cap N(W, V) \ne \emptyset$.