Duality results for Hardy Spaces on strongly convex domains with smooth boundary

Alekos Vidras, Department of Mathematics and Statistics,
University of Cyprus, Cyprus

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Formulation of the problem for bounded domains $\Omega \subset \mathbb{C}$ having Ahhfors regular boundaries.

The Hardy-Smirnov space $E^p(\Omega)$, $p > 1$ for a simply connected, bounded domain having Ahlfors regular boundary $\partial\Omega = \gamma$ consists of functions f holomorphic in Ω and satisfying $\int\limits_{\gamma_m} |f(z)|^p |dz| \leq A$ for some sequence of curves γ_m converging to γ (in a sense that the sequence $\{\gamma_m\}$ eventually surrounds any compact $V \subset \Omega$).

We want to describe precisely the space $(E^p(\Omega))'$, $p \geq 1$ in terms of holomorphic functions defined in $(\overline{\Omega})^c$ with some conditions on the boundary. Let us motivate this approach : Every element $\mathcal{F} \in (E^p(\Omega))'$ is represented by an element $g \in L^q(\partial\Omega)$. Thus for every polynomial $p(z)$ the action of the $\mathcal{F}(p)$ is described, via Fubini's Theorem,

$$\begin{aligned}\mathcal{F}(p) &= \int_{\partial\Omega} g(\zeta)p(\zeta)d\zeta = \int_{\partial\Omega} g(\zeta)(\frac{1}{2\pi i}\int_{\gamma}\frac{p(z)dz}{z-\zeta})d\zeta \\ &= \int_{\gamma} p(z)(\frac{1}{2\pi i}\int_{\partial\Omega}\frac{g(\zeta)d\zeta}{z-\zeta}dz), \qquad (1)\end{aligned}$$

whenever $\gamma \in (\overline{\Omega})^c$. Note that the value (1) remains constant whenever γ tends in a reasonable way to $\partial\Omega$. It is important to realize here that (1) can be viewed as an "boundary value " interpretation of the Kothe-Silva duality

Because the boundary of the domain is Ahlfors-regular, it is natural to expect that this constant is $-\int\limits_{\partial\Omega} p(\zeta)C^*_{ext,g}(\zeta)d\zeta$, where $C^*_{ext,g}(\zeta)$ is the external angular boundary value function of the Cauchy type integral $C_{ext,g}(z) = \int\limits_{\partial\Omega} \frac{g(\zeta)d\zeta}{\zeta - z}$, belonging to $L^q(\partial\Omega)$. Note that $C_{ext,g}(z)$ is holomorphic in $(\overline{\Omega})^c$ and $C_{ext,g}(\infty) = 0$. With this in mind we define the space

$$\mathcal{A}^p_0(\Omega^c), \ p > 1$$

, to be the space of **functions holomorphic in the open connected set $(\overline{\Omega})^c$, vanishing at infinity, having angular boundary values f^* on $\partial\Omega$ a.e and satisfying $\int\limits_{\partial\Omega} \frac{f(\zeta)d\zeta}{\zeta^n} = 0$, $n = 1, 2, \ldots$.**
While for $p = 1$ we require $f^* \circ \phi \in BMO(\mathbb{D})$, where ϕ is the normalized Riemmann mapping from the unit disk into Ω and in this case the space is denoted by

$$\mathcal{BA}^1_0(\Omega^c)$$

Theorem 1

Let $\Omega \subset \mathbb{C}$ be a bounded, simply connected domain whose boundary $\partial\Omega$ is Ahlfors-regular. Then for $p > 1$ one has that

$$(E^p(\Omega))' = \mathcal{A}_0^q(\Omega^c),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. For $p = 1$ one has that

$$(E^1(\Omega))' = \mathcal{B}\mathcal{A}_0^1(\Omega^c).$$

The idea of the proof (for the case $p > 1$) is simple : starting with the Plemelj-Sokhotskii formula $g(\zeta) = C^*_{int,g}(\zeta) - C^*_{ext,g}(\zeta)$, for almost all $\zeta \in \partial\Omega$ the first step is to prove that

$$\int_{\partial\Omega} C^*_{int,g}(\zeta) f(\zeta) = 0, \ \forall f \in E^p(\Omega). \qquad (2)$$

Combining (1), (2) and P-S formulas one has that

$$-\int_{\partial\Omega} C^*_{ext,g}(\zeta) f(\zeta) = \int_{\partial\Omega} g(\zeta) f(\zeta), \ \forall f \in E^p(\Omega). \qquad (3)$$

Proof of Theorem 1

Furthermore (2) implies that $g(\zeta) + C^*_{ext,g}(\zeta)$ defines the zero functional on $E^p(\Omega)$. In articular it means that

$$\int_{\partial\Omega} (g(\zeta) + C^*_{ext,g}(\zeta)\zeta^n d\zeta = 0, \ \forall n \in \mathbb{N}. \qquad (4)$$

Or equivalently

$$\int_{\partial\Omega} \frac{g(\zeta)}{\zeta - z} d\zeta = -\int_{\partial\Omega} \frac{C^*_{ext,g}(\zeta)}{\zeta - z} d\zeta, \ \forall z \in (\overline{\Omega})^c. \qquad (5)$$

The last equality at the boundary implies that

$$\int_{\partial\Omega} \frac{C^*_{ext,g}(\zeta))}{\zeta - z} d\zeta = 0 \ \forall z \in \Omega,$$

which is exactly the moment condition describing the space $\mathcal{A}^p_0(\Omega^c)$.

Duality for convex domains $\Omega \subset \mathbb{C}^n$

It is natural to expect that the duality of the space

$$H^p(\Omega) = \{f : \Omega \longrightarrow \mathbb{C} \text{ holomorphic and } : \limsup_{\epsilon \to 0} \int_{\partial\Omega} |f(\zeta - \epsilon\nu_\zeta)|^p d\sigma_\zeta < \infty\}, \quad (6)$$

where the bounded domain $\Omega = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \ : \varrho(z, \bar{z}) < 0\}$ is $\mathcal{C}^2$, and ν_ζ is the exterior normal, unit vector at $\zeta \in \partial\Omega$ and the integral is taken with respect to the hypersurface measure, is more complicated. The space $H^p(\Omega)$ is a vector space equipped with the norm $\|f\|_{p,\partial\Omega} = (\int_{\partial\Omega} |f(\zeta)|^p d\sigma_\zeta)^{\frac{1}{p}}$, where $f(\zeta) = \lim_{\epsilon \longrightarrow 0} f(\zeta - \epsilon\nu_\zeta)$, for almost all $\zeta \in \partial\Omega$. An alternative description for the definition of the Hardy space for the convex domain Ω given by (6) is the following :

$$\limsup_{0<r<1} \int_{\partial\Omega_r} |f(\zeta)|^p d\sigma_r(\zeta) < \infty, \quad (7)$$

where $\Omega_r = r\Omega$ is a homothety of Ω and $d\sigma_r$ is the surface measure on $\partial\Omega_r$.

It is rather straightforward to show that when the domain Ω is convex the description of Hardy spaces given by (6) and (7) describe the same space. However the multidimensional analogue of "boundary" Kothe-Silva duality given by (1) is rather more intricate. One main difficulty is that we do not have the universality of Cauchy integral representation formula that is available in one complex variable. One of the most known analogues to it is the Khenkin-Ramirez formula for pseudoconvex domains, however for domains known as regularly linearly convex domains $\Omega \subset \mathbb{C}^n$ there is an even better representation formula, known as Cachy-Leray-Fantappie formula, which has a number of advantages which are of great importance to us.

Definition

We call the domain $\Omega \subset \mathbb{C}^n$ **linearly convex**(or weakly linearly convex in [APS]) if for every $\zeta \in \partial\Omega$ there exists a complex hyperplane

$$\alpha = \{z \in \mathbb{C}^n : \alpha_1 z_1 + \cdots + \alpha_n z_n + \beta = 0\}$$

through ζ and does not intersect Ω.
We point out that the topological dimension of the hyperplane α is $2n - 2$. Note than any domain $\Omega \subset \mathbb{C}$ is linearly convex. In particular any convex set is linearly convex also, but the converse is not true. For example, if $\Omega_i \subset \mathbb{C}$, $i = 1, 2$, are bounded simply connected domains, then $\Omega_1 \times \Omega_2 \subset \mathbb{C}^2$ is linearly convex, but not convex (in general).

Exterior for the domain $\Omega \subset \mathbb{C}^n$

If $0 \in \Omega$, then its dual complement or generalized complement,(in the work of L.Aizenberg-A.Martineau it is called the conjugate set of Ω) is defined to be set

$$\widetilde{\Omega} = \{w \in \mathbb{C}^n : w_1 z_1 + \cdots + w_n z_n \neq 1, z \in \Omega\}$$

That is, it is the set of complex hyperplanes that do not intersect the domain Ω. Thus $0 \in \widetilde{\Omega}$. It is also a known fact that for the domain Ω the dual complement of its closure $\overline{\Omega}$ is the interior of the set $\widetilde{\Omega}$, that is, $\widetilde{\overline{\Omega}} = int(\widetilde{\Omega})$. In particular, when Ω has a smooth ($\mathcal{C}^2$) boundary then $\widetilde{\Omega} = int(\widetilde{\Omega}) \cup \partial\widetilde{\Omega}$. Furthermore, if the bounded domain Ω is convex then $\lambda\overline{\Omega} \subset \Omega$, for every $0 < \lambda < 1$. Thus the closed domain $\overline{\Omega}$ and the open domain $\widetilde{\overline{\Omega}}$ are starlike . Note again here, that for any domain $\Omega \subset \mathbb{C}$ one has that $\widetilde{\Omega} = \Omega^c$. Thus one can reinterpret (1) and the result of Theorem 1 thinking of functionals there to be represented by holomorphic functions defined on the dual complement.

Basic examples for the dual complements-Reinhardt domains

If Ω is a Reinhardt domain, then $F(\Omega) \subset \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \ x_i \geq 0\}$, $F(z_1, z_2, \dots, z_n) = (|z_1|, |z_2|, \dots, |z_n|)$. For any $B \subset \mathbb{R}^n_+$, its inverse image by F^{-1} is defined to be the set $F^{-1}(B) = \{(z_1, z_2, \dots, z_n) \in \mathbb{C}^n : F(z_1, z_2, \dots, z_n) \in B\}$. It is straightforward to verify that domain $\Omega \subset \mathbb{C}^n$ is Reinhardt if and only if $\Omega = F^{-1}(F(\Omega))$. Hence, any Reinhardt domain Ω is determined completely by its absolute image $F(\Omega)$. Thus we have the following definition

Let $\Omega \subset \mathbb{C}^n$ be a Reinhardt domain centered at the origin $0 \in \mathbb{C}^n$. We say that the point $(y_1, \dots, y_n) \in \widetilde{F(\Omega)} \subset \mathbb{R}^n_+$ if and only if $\sum_{i=1}^{n} x_i y_i < 1$ for every $(x_1, x_2, \dots, x_n) \in F(\Omega)$. Then the dual complement of Ω is the set $\widetilde{\Omega} = F^{-1}(\widetilde{F(\Omega)})$.

The last observation leads to the following :
For $r > 0$, $p \geq 1$ and $k_i \in \mathbb{R}^n_+ \setminus \{0\}$ fixed numbers, let

$$\Omega = \{z \in \mathbb{C}^n : \sum_{i=1}^{n} k_i |z_i|^p < r^p\},$$

be a Reinhardt domain centered at the origin. Then, for $q = \frac{p-1}{p}$,

$$\widetilde{\Omega} = \{\zeta \in \mathbb{C}^n : \sum_{i=1}^{n} (k_i)^{\frac{1}{1-p}} |\zeta_i|^q \leq \frac{1}{r^q}\}.$$

The previously known cases occur when $k_i = 1$, $(i = 1, \ldots, n)$. The appropriate case when $p = 1$ corresponds to the case when Ω is a hyper-cone, whose dual complement $\widetilde{\Omega}$ is the closed poly-disk. When $p = 2$, the appropriate domain Ω is a ball about the origin of radius r, whose dual complement is the closed ball about the origin of radius $\frac{1}{r}$. In the case $p = \infty$, the dual complement $\widetilde{\Omega}$ is the closed hypercone. So the above gives new results for $p > 2$ or $k \neq 1$.

Cauchy-Leray-Fantappie formula for strictly convex domains

Let $\Omega = \{z \in \mathbb{C}^n : \varrho(z, \bar{z}) < 0\}$ be bounded convex domain with smooth ($\mathcal{C}^2$ at least) boundary. If $f \in L^p(\partial\Omega)$, $p > 1$, then the non-tangential boundary value function of the Cauchy-Fantappiè integral of the function f

$$CF(f)(z) = \frac{(n-1)!}{(2\pi)^n} \int\limits_{\partial\Omega} \frac{f(\zeta)\partial_\zeta \varrho \wedge (\partial_\zeta \partial_{\bar{\zeta}} \varrho)^{n-1}}{< \nabla_\zeta \varrho, \zeta - z >^n}, \ z \in \Omega, \quad (8)$$

where $\partial_\zeta \varrho = \partial\varrho(\zeta, \bar{\zeta})$, $\partial_\zeta \partial_{\bar{\zeta}} \varrho = \partial\bar{\partial}\varrho(\zeta, \bar{\zeta})$ $\nabla_\zeta \varrho = (\partial_{\zeta_1} \varrho, \dots, \partial_{\zeta_n} \varrho)$, is given by the Sokhotskii-Plemelj formula (this follows from results contained in [**?**]) that

$$CF(f)(\zeta_0) = \frac{1}{2} f(\zeta_0) + c_n \int\limits_{\partial\Omega} \frac{f(\zeta)\partial\varrho_\zeta \wedge (\partial_\zeta \partial_{\bar{\zeta}} \varrho)^{n-1}}{< \nabla_\zeta \varrho, \zeta - \zeta_0 >^n}, \quad (9)$$

valid for almost all $\zeta_0 \in \partial\Omega$ and where $c_n = \frac{(n-1)!}{(2\pi i)^n}$.

$$f(\zeta_0) = CF(f)(\zeta_0), \ \forall \zeta_0 \in \partial\Omega$$

whenever $f \in A(\Omega)$

As usual,

$$\int_{\partial\Omega} \frac{f(\zeta)\partial_\zeta \varrho \wedge (\partial_\zeta \partial_{\bar\zeta} \varrho)^{n-1}}{< \nabla_\zeta \varrho, \zeta - \zeta_0 >^n} = \lim_{\epsilon \longrightarrow 0} \int_{\partial\Omega_{\varrho,\epsilon}} \frac{f(\zeta)\partial_\zeta \varrho \wedge (\partial_\zeta \partial_{\bar\zeta} \varrho)^{n-1}}{< \nabla_\zeta \varrho, \zeta - \zeta_0 >^n},$$

where for $\epsilon > 0$ the set $\partial\Omega_{\varrho,\epsilon}$ is defined by

$$\partial\Omega_{\varrho,\epsilon} = \{\zeta \in \partial\Omega : \ | < \nabla\varrho(\zeta), \zeta - \zeta_0 >^n | \geq \epsilon\}.$$

Multidimensional analogue of Aizenberg-Martineau for Kothe-Silva duality

If Ω is convex, then $\Omega = \lim_{m \longrightarrow \infty} \Omega_m$, where $\overline{\Omega}_m \subset \Omega_{m+1}$, $0 \in \Omega_m$ and the domains

$$\Omega_m = \{z \in \mathbb{C}^n : \ \varrho_m(z, \bar{z}) < 0\}, \ m = 1, 2 \ldots$$

are bounded with functions ϱ_m being $\mathcal{C}^2$ and convex in a neighborhood of the domain $\overline{\Omega}_m$. It is obvious that $\widetilde{\Omega} \subset \widetilde{\Omega}_m$ and $\widetilde{\Omega}_{m+k} \subset \widetilde{\Omega}_m$, $m = 1, 2, 3 \ldots$, for every $k = 1, 2 \ldots$.
Now, for every $\zeta \in \partial\Omega_m$ one can consider (locally) the hyperplane

$$\{z \in \mathbb{C}^n : \sum_{i=1}^{n} (\zeta_i - z_i) \varrho'_{m\zeta_i}(\zeta) = 0\} = \{z \in \mathbb{C}^n : \sum_{i=1}^{n} z_i \frac{\varrho'_{m\zeta_i}}{< \nabla_\zeta \varrho, \zeta >} = 1\}$$

which does not intersect the domain Ω_m for any $z \in \Omega_m$. Thus the vector

$$\tau(\varrho_m) = (\tau_1(\varrho_m), \ldots, \tau_n(\varrho_m)) = (\frac{\varrho'_{m\zeta_1}}{< \nabla_\zeta \varrho, \zeta >}, \ldots, \frac{\varrho'_{m\zeta_n}}{< \nabla_\zeta \varrho, \zeta >}) \in \widetilde{\Omega}_m. \quad (10)$$

Hence

$$\tau(\varrho_m) \in \widetilde{\Omega}_l, \ \forall m > l. \quad (11)$$

We observe that the function

$$l(z,w) = \frac{1}{1 - z_1 w_1 - \cdots - z_n w_n},\ (z,w) \in \Omega \times \widetilde{\Omega},$$

is holomorphic for $z \in \Omega$ and for $w \in U$, where U is an open set containing $\widetilde{\Omega}$.

Let $z \in \Omega$, then there exists $m_0 > 0$ so that $z \in \Omega_m$ for every $m \geq m_0$. When $w = (\tau_1(\varrho_m), \ldots, \tau_n(\varrho_m)) \in \widetilde{\Omega}_m \subset \widetilde{\Omega}_{m_0}$, the Cauchy-Fantappiè integral representation formula for $f \in H(\Omega)$ implies

$$\begin{aligned} f(z) &= \int_{\partial\Omega_m} \frac{f(\zeta)\omega(\zeta,\tau(\varrho_m))}{< \zeta - z, \nabla_\zeta \varrho >^n} \\ &= \int_{\partial\Omega_m} \frac{f(\zeta)\omega(\zeta,\tau(\varrho_m))}{(1 - \sum_{i=1}^{n} z_i \tau_i(\varrho_m))^n}, \qquad (12) \end{aligned}$$

for $z \in \Omega_m$, $\zeta \in \partial\Omega_m$ and where

$$\begin{aligned} \omega(z,w) &= \left(\frac{1}{2\pi i}\right)^n \sum_{k=1}^{n} (-1)^{k-1} w_k dw[k] \wedge dz, \qquad (13) \\ dw[k] &= dw_1 \wedge \cdots \wedge dw_{k-1} \wedge dw_{k+1} \wedge \cdots \wedge dw_n,\ dz = dz_1 \wedge \cdots \wedge dz_n, \end{aligned}$$

Any analytic functional F, can be expressed (Hahn-Banach Theorem) through a measure compactly supported in Ω. That is, there exists a compactly supported measure μ_Ω, so that the support $\mathcal{A}$ of μ_Ω is contained in Ω_m for every $m \geq n_0$, when n_0 is sufficiently large, and thus

$$\begin{aligned} F(f) &= \int\limits_{\Omega_m} f(z) d\mu_\Omega = \int\limits_{\Omega_m} \left(\int\limits_{\partial\Omega_{m'}} \frac{f(\zeta)\omega(\zeta, \tau(\varrho_{m'}))}{(1 - \sum_{i=1}^{n} z_i \tau_i(\varrho_{m'}))^n} \right) d\mu_\Omega \\ &= \int\limits_{\partial\Omega_{m'}} f(\zeta) \left(\int\limits_{\Omega_m} \frac{d\mu_\Omega}{(1 - \sum_{i=1}^{n} z_i \tau_i(\varrho_{m'}))^n} \right) \omega(\zeta, \tau(\varrho_{m'})) \\ &= \int\limits_{\partial\Omega_{m'}} f(\zeta) \phi(\tau(\varrho_{m'})) \omega(\zeta, \tau(\varrho_{m'})), \end{aligned} \tag{14}$$

for any $\Omega_{m'}$ containing Ω_m (that is, for any index $m' > m$) and where $f \in H(\Omega)$.

Observe that

$$\phi(\tau(\varrho_{m'})) = \int_{\Omega_m} \frac{d\mu_\Omega}{(1-\sum_{i=1}^{n} z_i\tau_i(\varrho_{m'}))^n} = \int_{\Omega_{m'}} \frac{d\mu_\Omega}{(1-\sum_{i=1}^{n} z_i\tau_i(\varrho_{m'}))^n}$$
$$= F\left(\frac{1}{(1-\sum_{i=1}^{n} z_i\tau_i(\varrho_{m'}))^n}\right)$$

belongs to $H(\widetilde{\Omega})$ because of (10) and the fact that the the set $\mathcal{A} = \textit{support of } \mu_\Omega \subset \Omega_m \subset \Omega_{m'}$. The index m depends on the function ϕ, which is holomorphic on the larger compact set $\widetilde{\Omega} \subset \widetilde{\Omega}_m$. The integral in (14) does not depend on the choice of m. The converse direction one considers the function $\phi \in H(\widetilde{\Omega})$. Let $w \in B(o,r) \subset \widetilde{\Omega}$ be fixed, for an appropriate $r > 0$. For this w the function $f_w(z) = \frac{1}{(1-z_1w_1-\cdots-z_nw_n)^n}$, $z \in \Omega$ is analytic. Then, using the Cauchy-Fantappie formula for the ball $B(o,r)$ and the function ϕ one has that $F_\phi(f_w) = \phi(w)$.

Duality for Hardy spaces

Theorem 2 : Let $\Omega = \{z \in \mathbb{C}^n : \varrho(z, \bar{z}) < 0\}$, where $\varrho \in \mathcal{C}^3(\overline{\Omega})$ is its defining function, be bounded, strictly convex domain. If $0 \in \Omega$, then

$$(H^p(\Omega))' = H^q(\widetilde{\Omega}), \tag{15}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Furthermore, the isomorphism is realized

$$F(f) = F_\phi(f) = \int_{\partial\Omega} \phi(w) f(z) \omega(z, w), \tag{16}$$

where $\phi \in H^q(\widetilde{\Omega})$, $f \in H^p(\Omega)$ and ω, w are defined by (13) and (10) respectively.

First, we recall the basic facts to be used below : 1^0) $\widetilde{(r\Omega)} = \frac{1}{r}\widetilde{\Omega}$, 2^0) For linearly convex (in particular for a bounded convex) domain Ω with smooth boundary we have that

$$\partial\widetilde{\Omega} = \{w \in \mathbb{C}^n : w_j \text{ are defined by } (10), z \in \partial\Omega\}.$$

Thus the boundary $\partial\widetilde{\Omega}$ is smooth enough, provided $\partial\Omega$ is also smooth. Consider the space $L^q(\partial\widetilde{\Omega})$. Then the space $H^q(\widetilde{\Omega})$ is closed subspace of $L^q(\partial\widetilde{\Omega})$ with respect to the L^q-norm. Thus, for every element $F \in \left(H^q(\widetilde{\Omega})\right)'$ there exists a function $g \in L^p(\partial\widetilde{\Omega})$ such that

$$F(f) = \int\limits_{\partial\widetilde{\Omega}_z} f(z)\overline{g(z)}\omega(w(z), z),$$

where the coordinates for w defined on the domain $\widetilde{\Omega}$ are given by formulas similar to the coordinates given in (10). Furthermore, we remark that the form $\omega(w(z), z)$ is non-degenerate on $\partial\widetilde{\Omega}_z$ (Lemma 5.12 in the book by A-D).

Now, recapitulating on the definition of the functional F, one has that

$$\begin{aligned} F(f) &= \int\limits_{\partial\widetilde{\Omega}_z} f(z)\overline{g(z)}\omega(w(z),z) \\ &= \int\limits_{\partial\widetilde{\Omega}_z} \overline{g(z)} \lim_{r\longrightarrow 1}\left(\int\limits_{\partial\widetilde{\Omega}_{r,\eta}} \frac{f(\eta)\omega(w(\eta)),\eta)}{< 1-\omega(w(\eta)),z>^n}\right)\omega(w(z),z), \end{aligned}$$

where $0<r<1$, $\widetilde{\Omega}_{r,\eta}=r\widetilde{\Omega}_\eta$-homothety. Taking the limit outside the integral and changing the order of integration leads to

$$F(f) = \lim_{r\longrightarrow 1}\int\limits_{\partial\widetilde{\Omega}_{r,\eta}} f(\eta)\left(\int\limits_{\partial\widetilde{\Omega}_z} \frac{\overline{g(z)}\omega(w(z)),z)}{< 1-\omega(w(\eta)),z>^n}\right)\omega(w(\eta),\eta).$$

The convexity of the domain Ω (hence its linear convexity) implies that $\widetilde{\left(\widetilde{\Omega}\right)}=\Omega$. Thus, in the inner integral , we make a change of variables

$$w:\zeta\in\Omega=\widetilde{\left(\widetilde{\Omega}\right)}\longrightarrow w(\zeta)\in\widetilde{\Omega}$$

and deduce that

$$F(f) = \lim_{r \longrightarrow 1} \int\limits_{\partial \widetilde{\Omega}_{r,\eta}} f(\eta) \left(\int\limits_{\partial \widetilde{\widetilde{\Omega}}_{\zeta}} \frac{G(w)\omega(\widetilde{w}, w)}{< 1 - \omega(w(\eta), w >^n} \right) \omega(w(\eta), \eta), \tag{17}$$

where $\widetilde{w}$ is a vector whose components are obtained after the change of variables $w(\zeta)$. The form $\omega(\widetilde{w}, w)$ can be represented as function multiplied by form in w as in Cauchy-Fantappiè formula, since the last one is non-degenerate on $\partial \widetilde{\left(\widetilde{\Omega}_{\zeta}\right)}$. Now we apply a result from Kerzman-Stein, from where it follows that the integral of the Cauchy-Fantappiè type acts as a continuous linear operator from L^p-space on the boundary of strictly pseudo-convex domain with $\mathcal{C}^3$ boundary into H^p space in the domain for $p > 1$ (though the above result is formulated for domains with $\mathcal{C}^\infty$ boundary, in its proof only the fact that it is $\mathcal{C}^3$ is used).

Furthermore, since the domain $\widetilde{\left(\widetilde{\Omega}\right)} = \Omega$ is strictly convex it is also strictly pseudoconvex. Finally, the inner integral in (17) is a function from $H^p(\widetilde{\Omega}_{r,\eta})$. Take the limit $r \longrightarrow 1$ in (17) and use 1^0). The continuity of the above operator yields :

$$F(f) = \int\limits_{\partial\Omega_\eta} f(\eta)\phi(w)\omega(w(\eta),\eta),$$

where $\phi(w) \in H^p(\widetilde{\widetilde{\Omega}})$ denotes the integral in parenthesis in (17). Thus $\left(H^q(\widetilde{\Omega})\right)' = H^p(\Omega)$, since $\widetilde{\left(\widetilde{\Omega}\right)} = \Omega$. Therefore $\left(\left(H^q(\widetilde{\Omega})\right)'\right)' = (H^p(\Omega))'$. These spaces are reflexive, thus (15) follows.