Schatten-type Classes of Operators on Kaplansky–Hilbert Modules

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Kaplansky−Hilbert module, which is an object like a Hilbert space except that the inner product is not scalar-valued but takes its values in a commutative C^*-algebra Λ which is an order complete vector lattice, was introduced by I. Kaplansky [1]. Such a C^*-algebra is often called a Stone algebra or a commutative AW^*-algebra. I. Kaplansky proved some deep and elegant results for such structures, thereby showing that they have many properties similar to those of the Hilbert spaces.

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Cyclically compact sets and operators in lattice-normed spaces were introduced by A. G. Kusraev and a preliminary study of this notions was initiated. Cyclical compactness is the Boolean-valued interpretation of compactness and it also deserves an independent study. In [5] a general form of cyclically compact operators in Kaplansky–Hilbert modules, which is similar to the Schmidt representation of compact operators on Hilbert spaces, as well as a variant of the Fredholm alternative for cyclically compact operators, were also given with Boolean-valued techniques. Thus, the natural problem arises to investigate the class of cyclically compact operators in more details.

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- **Definition.** Let Λ be a Stone algebra and X be a Λ-module. The mapping $\langle\cdot \mid \cdot\rangle : X \times X \to \Lambda$ is a Λ-*valued inner product*, if for all x, y, z in X and a in Λ the following are satisfied:
  (1) $\langle x \mid x\rangle \geq 0$; $\langle x \mid x\rangle = 0 \Leftrightarrow x = 0$;
  (2) $\langle x \mid y\rangle = \langle y \mid x\rangle^*$;
  (3) $\langle ax \mid y\rangle = a\langle x \mid y\rangle$;
  (4) $\langle x + y \mid z\rangle = \langle x \mid z\rangle + \langle y \mid z\rangle$.
- The Cauchy–Bunyakovskiĭ–Schwarz inequality is valid

$$|\langle x \mid y\rangle| \leq |x|\,|y|.$$

- We can introduce the norms in X by the formulas

$$|x| := \sqrt{\langle x \mid x\rangle}, \qquad |||x||| := \|\langle x \mid x\rangle\|^{\frac{1}{2}}.$$

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- A *Kaplansky–Hilbert module* or an AW^*-*module* over Λ is a C^*-module satisfing the following two properties:
  (1) let $x \in X$, and let $(e_\xi)_{\xi \in \Xi}$ be a partition of unity in $P(\Lambda)$ with $e_\xi x = 0$ for all $\xi \in \Xi$; then $x = 0$;
  (2) let $(x_\xi)_{\xi \in \Xi}$ be a norm-bounded family in X, and let $(e_\xi)_{\xi \in \Xi}$ be a partition of unity in $P(\Lambda)$; then there exists an element $x \in X$ such that $e_\xi x = e_\xi x_\xi$ for all $\xi \in \Xi$.
- It follows from (1) that the element x of (2) is unique; we shall write $x = \mathrm{mix}_{\xi \in \Xi}(e_\xi x_\xi)$.
- Throughout the rest of this talk, the letters X and Y denote Kaplansky–Hilbert modules over Λ.

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Definition. A subset $\mathscr{E}$ of X is said to be *orthonormal* (*projection orthonormal*) if

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An orthonormal (projection orthonormal) set $\mathscr{E} \subset X$ is a *basis* (*projection basis*) for X provided that

(3) the condition $(\forall e \in \mathscr{E})\ \langle x \mid e \rangle = 0$ implies $x = 0$.

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Let Q be an extremal compact space and E be a normed space. Vector-functions $u : \operatorname{dom} u \to E$ and $v : \operatorname{dom} v \to E$ are equivalent

$$\text{if } u(t) = v(t) \text{ whenever } t \in \operatorname{dom} u \cap \operatorname{dom} v$$

where $\operatorname{dom} u$ and $\operatorname{dom} v$ are comeager subsets of Q. Let $C_{\#}(Q, E)$ be the set of the equivalence classes of bounded continuous vector-functions u. The set $C_{\#}(Q, E)$ is endowed, in a natural way, with the structure of a module over $C(Q)$. Note that each bounded continuous function $f : \operatorname{dom} f \to \mathbb{R}$ admits a unique continuous extention $\overline{f} : Q \to \mathbb{R}$. Therefore, we can introduce the vector norm $\left|\cdot\right| : C_{\#}(Q, E) \to C(Q)$ by the formula

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Let H be a Hilbert space and $\langle\cdot,\cdot\rangle$ be the inner product of H. Then we can introduce some $C(Q)$-valued inner product in $C_{\#}(Q,H)$ as follows. Let $\widetilde{u},\widetilde{v}\in C_{\#}(Q,H)$. The function

$$q\mapsto\langle u(q),v(q)\rangle\ \ (q\in\operatorname{dom}u\cap\operatorname{dom}v)$$

is continuous and admits a unique continuous extention $z\in C(Q)$. If x and y are the equivalence classes containing vector-functions u and v then assign

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Theorem. [5, 7.4.8.(1)] Suppose that Q is an extremal compact space, and H is a Hilbert space of dimension λ. The space $C_{\#}(Q, H)$ is a λ-homogeneous Kaplansky–Hilbert module over the algebra $C(Q)$.

The following result on functional representation of Kaplansky–Hilbert modules is one of the main tools of our investigation.

Theorem.[5, Theorem 7.4.12.] *To each Kaplansky–Hilbert module X over Λ there is a family of nonempty extremal compact spaces $(Q_\gamma)_{\gamma\in\Gamma}$ with Γ a set of cardinals, such that Q_γ is γ-stable for all $\gamma \in \Gamma$ and the following unitary equivalence holds:*

$$X \simeq \sum_{\gamma\in\Gamma}^{\oplus} C_{\#}\left(Q_\gamma, \ell_2(\gamma)\right).$$

If some family $(P_\delta)_{\delta\in\Delta}$ of extremal compact spaces satisfies the above conditions then $\Gamma = \Delta$ and P_γ is homeomorphic with Q_γ for all $\gamma \in \Gamma$.

Let B be a complete Boolean algebra. Denote by $\mathrm{Prt}(B)$ the set of sequences $\nu : \mathbb{N} \to B$ which are partitions of unity in B. For $\nu_1, \nu_2 \in \mathrm{Prt}(B)$, the formula $\nu_1 \ll \nu_2$ abbreviates the following assertion:

$$\text{if } m, n \in \mathbb{N} \text{ and } \nu_1(m) \wedge \nu_2(n) \neq 0_B \text{ then } m < n.$$

Given a mix-complete subset $K \subset X$, a sequence $s : \mathbb{N} \to K$, and a partition $\nu \in \mathrm{Prt}(B)$, put $s_\nu := \mathrm{mix}_{n \in \mathbb{N}} (\nu(n) s(n))$. A *cyclic subsequence* of $s : \mathbb{N} \to K$ is any sequence of the form $(s_{\nu_k})_{k \in \mathbb{N}}$, where $(\nu_k)_{k \in \mathbb{N}} \subset \mathrm{Prt}(B)$ and $(\forall k \in \mathbb{N})\ \nu_k \ll \nu_{k+1}$. A subset $C \subset X$ is said to be *cyclically compact* if C is mix-complete and every sequence in C has a cyclic subsequence that converges (in norm) to some element of C. A subset in X is called *relatively cyclically compact* if it is contained in a cyclically compact set.

Cyclically Compact Sets in Kaplansky–Hilbert Modules

Let B be a complete Boolean algebra. Denote by $\mathrm{Prt}(B)$ the set of sequences $\nu : \mathbb{N} \to B$ which are partitions of unity in B. For $\nu_1, \nu_2 \in \mathrm{Prt}(B)$, the formula $\nu_1 \ll \nu_2$ abbreviates the following assertion:

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Given a mix-complete subset $K \subset X$, a sequence $s : \mathbb{N} \to K$, and a partition $\nu \in \mathrm{Prt}(B)$, put $s_\nu := \mathrm{mix}_{n \in \mathbb{N}} (\nu(n)s(n))$. A *cyclic subsequence* of $s : \mathbb{N} \to K$ is any sequence of the form $(s_{\nu_k})_{k \in \mathbb{N}}$, where $(\nu_k)_{k \in \mathbb{N}} \subset \mathrm{Prt}(B)$ and $(\forall k \in \mathbb{N})\ \nu_k \ll \nu_{k+1}$. A subset $C \subset X$ is said to be *cyclically compact* if C is mix-complete and every sequence in C has a cyclic subsequence that converges (in norm) to some element of C. A subset in X is called *relatively cyclically compact* if it is contained in a cyclically compact set.

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Cyclically Compact Sets in Kaplansky–Hilbert Modules

- Define $K(q) := \{x(q) : \widetilde{x} \in K, q \in \mathrm{dom}(x)\}$ $(q \in Q, K \subset C_{\#}(Q, H))$.
- **Proposition.** *Let K be a relatively cyclically compact subset of $C_{\#}(Q, H)$. Then there exists a comeager set $Q_0 \subset Q$ such that $K(q)$ is precompact in H for all $q \in Q_0$.*
- **Proposition.** *Let K be a mix-complete subset of $C_{\#}(Q, H)$ and $Q_0 \subset Q$ be a comeager subset. If K_q is a compact set in H for $q \in Q_0$ and*

$$K \subset \{\widetilde{x} \in C_{\#}(Q, H) : x(q) \in K_q\,(\forall q \in \mathrm{dom}(x) \cap Q_0)\},$$

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Operators on Kaplansky–Hilbert Modules

- Let $B_\Lambda(X, Y)$ denote the set of all continuous Λ-linear operators from X into Y. For brevity, $B_\Lambda(X, X)$ will be denoted by $B_\Lambda(X)$.
- We call a Λ-linear operator $T^* : Y \to X$ the *adjoint* of Λ-linear operator $T : X \to Y$ if $\langle Tx \mid y \rangle = \langle x \mid T^* y \rangle$ for all x and y.
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Cyclically Compact Operators on Kaplansky–Hilbert Modules

- An operator $T \in B_\Lambda(X, Y)$ is called *cyclically compact* if the image $T(C)$ of any bounded subset $C \subset X$ is relatively cyclically compact in Y.
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Theorem.[5, Theorem 8.5.6.] *Let T in $\mathscr{K}(X,Y)$. There are orthonormal families $(e_k)_{k\in\mathbb{N}}$ in X, $(f_k)_{k\in\mathbb{N}}$ in Y, and a family $(\mu_k)_{k\in\mathbb{N}}$ in Λ such that the following hold:*

(1) $\mu_{k+1} \leq \mu_k$ $(k \in \mathbb{N})$ *and* $\inf \mu_k = 0$;

(2) *the representation is valid*

$$T = bo\text{-}\sum_{k=1}^{\infty} \mu_k \theta_{e_k,f_k}.$$

Theorem. *Let T be in $B_\Lambda(X)$ and Θ denote the set of all finite subsets of projection basis $\mathscr{E}$. Then the following statements are equivalent:*

(i) *T is a cyclically compact operator on X;*

(ii) *for every projection basis $\mathscr{E}$ in X, the net $\big(\big|T(I-P_F)\big|\big)_{F\in\Theta}$ o-converges to 0, where $P_F := \sum_{e\in F}\theta_{e,e}$;*

(iii) *for every projection basis $\mathscr{E}$ in X, one has* $\inf\left\{\sup\left\{\big|Te\big| : \ e\in F^c\right\} : \ F\in\Theta\right\}=0;$

(iv) *for every projection basis $\mathscr{E}$ in X, one has* $\inf\{\sup\{|\langle Te \mid e\rangle| : \ e\in F^c\} : \ F\in\Theta\}=0.$

- **Definition.** The *Hilbert–Schmidt class* $\mathscr{S}_2(X,Y)$ consists of cyclically compact operators T such that $(\mu_k^2)_{k\in\mathbb{N}}$ is o-summable in Λ. Put

$$v_2(T) := \left(o\text{-}\sum_{k\in\mathbb{N}} \mu_k^2 \right)^{1/2}.$$

The operators of the class $\mathscr{S}_2(X,Y)$ are called *Hilbert–Schmidt operators*.

- **Proposition.** *$(\mathscr{S}_2(X,Y), \langle\cdot,\cdot\rangle)$ is a Kaplansky–Hilbert module over Λ and the following equality holds,*

$$\left|T\right| \leq v_2(T) \qquad (T \in \mathscr{S}_2(X,Y))$$

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- **Definition.** The *trace class* $\mathscr{S}_1(X, Y)$ consists of cyclically compact operators T such that $(\mu_k)_{k\in\mathbb{N}}$ is o-summable in Λ. We put

$$v_1(T) := o\text{-}\sum_{k\in\mathbb{N}} \mu_k.$$

  The operators of class $\mathscr{S}_1(X, Y)$ are called *trace class operators*.

- **Proposition.** $(\mathscr{S}_1(X, Y), v_1(\cdot))$ *is a Banach–Kantorovich space and the following equality holds,*

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The Trace Class

Definition. For $T \in \mathscr{S}_1(X)$ define the *trace* of T by

$$\operatorname{tr}(T) := o\text{-}\sum_{e \in \mathscr{E}} \langle Te \mid e \rangle$$

where $\mathscr{E}$ is a projection basis of X.

- Denote by $\mathscr{X}^*$ the set of all Λ-linear operator $\eta : \mathscr{X} \to \Lambda$ such that there exists $c \in \Lambda$ with $|\eta(x)| \leq c \left| x \right|$ $(x \in \mathscr{X})$.
- **Theorem.** *If* $\varphi : \mathscr{S}_1(Y, X) \to \mathscr{K}(X, Y)^*$ *is defined by*

$$\varphi(T)(A) = \mathrm{tr}(TA)$$

for all $A \in \mathscr{K}(X, Y)$ *and* $T \in \mathscr{S}_1(Y, X)$*, then* φ *satisfies the following properties:*

  (i) φ *is a bijection* Λ*-linear operator from* $\mathscr{S}_1(Y, X)$ *to* $\mathscr{K}(X, Y)^*$*;*

  (ii) $v_1(T) = \left|\varphi(T)\right|$ $(T \in \mathscr{S}_1(Y, X))$.

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(ii) $v_1(T) = \left|\varphi(T)\right|$ $(T \in \mathscr{S}_1(Y, X))$.

- **Theorem.** $\psi : (B_\Lambda(X,Y), \|\cdot\|) \to (\mathscr{S}_1(Y,X)^*, \|\cdot\|_1)$ *is defined by*

$$\psi(L)(T) = \mathrm{tr}(TL)$$

for all $L \in B_\Lambda(X,Y)$ *and* $T \in \mathscr{S}_1(Y,X)$. *Then* ψ *satisfies the following properties:*

(i) ψ *is a bijection* Λ*-linear operator from* $B_\Lambda(X,Y)$ *to* $\mathscr{S}_1(Y,X)^*$*;*

(ii) $\|L\| = \|\psi(L)\|_1$ $(L \in B_\Lambda(X,Y))$.

If a, b are in Λ, then the notation $a \prec b$ means that $a \leq b$ and $0 \notin \mathrm{Sp}(b - a)$. Let λ, p be in Λ with $\lambda \geq 0$ and $0 \prec p$. It is well-known that there exists an extremal compact space Q such that Λ is $*$-isomorphic to $C(Q)$. So, λ^p in Λ can be defined as

$$(\lambda^p)(t) := (\lambda(t))^{p(t)} \ \ (t \in Q).$$

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$$(\lambda^p)(t) := (\lambda(t))^{p(t)} \;\; (t \in Q).$$

Definition. Let $\mathbf{1} \leq p \in \Lambda$. The symbol $\mathscr{S}_p(X, Y)$ denotes the set of all cyclically compact operators T such that $(\mu_k^p)_{k\in\mathbb{N}}$ is o-summable in Λ. We put

$$v_p(T) := \left(o\text{-}\sum_{k\in\mathbb{N}} \mu_k^p \right)^{\frac{\mathbf{1}}{p}}.$$

Proposition. *Let T be a cyclically compact operator from X to Y and $\mathbf{1} \leq p \in \Lambda$. Then the following statements are equivalent:*

(i) *T is in $\mathscr{S}_p(X, Y)$;*

(ii) *$(|\langle Te_\alpha \mid f_\alpha \rangle|^p)_{\alpha \in \mathscr{A}}$ is o-summable for all projection orthonormal subsets $(e_\alpha)_{\alpha \in \mathscr{A}}$ and $(f_\alpha)_{\alpha \in \mathscr{A}}$ in X and Y, respectively.*

In this case, for all $T \in \mathscr{S}_p(X, Y)$, the following equality holds

$$v_p(T) = \max \left\{ \left(o\text{-}\sum_{\alpha \in \mathscr{A}} |\langle Te_\alpha \mid f_\alpha \rangle|^p \right)^{\frac{1}{p}} \in \Lambda \right\}$$

Proposition.[6, Proposition 1.3] *If the mapping* $\sigma : X \times Y \to \Lambda$ *satisfies the properties*

(i) $\sigma(\lambda x_1 + \mu x_2, y) = \lambda\sigma(x_1, y) + \mu\sigma(x_2, y)$ $(x_1, x_2 \in X,\ y \in Y,\ \lambda, \mu \in \Lambda)$;

(ii) $\sigma(x, \lambda y_1 + \mu y_2) = \lambda^*\sigma(x, y_1) + \mu^*\sigma(x, y_2)$ $(x \in X,\ y_1, y_2 \in Y,\ \lambda, \mu \in \Lambda)$;

(iii) *There exists some* $\lambda \in \Lambda_+$ *such that* $|\sigma(x, y)| \leq \lambda \big|x\big| \, \big|y\big|$ $(x \in X,\ y \in Y)$,

then there exists a unique $A \in B_\Lambda(X, Y)$ *such that* $\big|A\big| \leq \lambda$ *and* $\sigma(x, y) = \langle Ax \mid y \rangle$.

Theorem. *Let* $\mathbf{1} \prec p, q \in \Lambda$ *and* $\mathbf{1}/p + \mathbf{1}/q = \mathbf{1}$. *If* $\phi : (\mathscr{S}_p(X), v_p(\cdot)) \to (\mathscr{S}_q(X)^*, \left|\cdot\right|_q)$ *is defined by*

$$\phi(T)(S) = \mathrm{tr}(ST)$$

for all $T \in \mathscr{S}_p(X)$ *and* $S \in \mathscr{S}_q(X)$, *then* ϕ *satisfies the following properties:*

(i) ϕ *is a bijection* Λ*-linear operator from* $\mathscr{S}_p(X)$ *to* $\mathscr{S}_q(X)^*$;

(ii) $v_p(T) = \left|\phi(T)\right|_q \ (T \in \mathscr{S}_p(X))$.

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